LiePRing
—
A GAP4 Package

Version 2.9.1

by

Bettina Eick (Braunschweig)
and
Michael Vaughan-Lee (Oxford)

Contents

1 Preamble

Abstract: This package gives access to the database of Lie p-rings of order at most p^7 as determined by Mike Newman, Eamonn O'Brien and Michael Vaughan-Lee, see [NOVL03] and [OVL05], and it provides some functionality to work with these Lie p-rings.

How to cite this package: If you use the LiePRing package, then please cite it as: *Bettina Eick and Michael Vaughan-Lee, LiePRing – A GAP Package for computing with nilpotent Lie rings of prime-power order (2014), see*

`https://www.gap-system.org/Packages/liepring.html`

Acknowlegdements: *The Lazard correspondence induces a one-to-one correspondence between the Lie p-rings of order p^n and class less than p and the p-groups of order p^n and class less than p. This package provides a function to evaluate this correspondence; this function has been implemented and given to us by Willem de Graaf.*

2 Lie p-rings

In this preliminary chapter we recall some of theoretic background of Lie rings and Lie p-rings. We refer to Chapter 5 in [Khu98] for some further details. Throughout we assume that p stands for a rational prime.

A Lie ring L is an additive abelian group with a multiplication that is alternating, bilinear and satisfies the Jacobi identity. We denote the product of two elements g and h of L with gh.

A subset $I \subseteq L$ is an ideal in the Lie ring L if it is a subgroup of the additive group of L and it satisfies $al \in I$ for all $a \in I$ and $l \in L$. As the multiplication in L is alternating, it follows that $la \in I$ for all $l \in L$ and $a \in I$. Note that if I and J are ideals in L, then $I + J = \{a + b \mid a \in I, b \in J\}$ and $IJ = \langle ab \mid a \in I, b \in J\rangle_+$ are ideals in L.

A subset $U \subseteq L$ is a subring of the Lie ring L if U is a Lie ring with respect to the addition and the multiplication of L. Every ideal in L is also a subring of L. As usual, for an ideal I in L the quotient L/I has the structure of a Lie ring, but this does not hold for subrings.

The lower central series of the Lie ring L is the series of ideals $L = \gamma_1(L) \geq \gamma_2(L) \geq \ldots$ defined by $\gamma_i(L) = \gamma_{i-1}(L)L$. We say that L is nilpotent if there exists a natural number c with $\gamma_{c+1}(L) = \{0\}$. The smallest natural number with this property is the class of L.

The notion of nilpotence now allows to state the central definition of this package. A **Lie p-ring** *is a Lie ring that is nilpotent and has p^n elements for some natural number n.*

Every finite dimensional Lie algebra over a field with p elements is an example for a Lie ring with p^n elements. Note that there exist non-nilpotent Lie algebras of this type: the Lie algebra consisting of all $n \times n$ matrices with trace 0 and $n \geq 3$ is an example. Thus not every Lie ring with p^n elements is nilpotent. (In contrast to the group case, where every group with p^n elements is nilpotent!)

For a Lie p-ring L we define the series $L = \lambda_1(L) \geq \lambda_2(L) \geq \ldots$ via $\lambda_{i+1}(L) = \lambda_i(L)L + p\lambda_i(L)$. This series is the lower exponent-p central series of L. Its length is the p-class of L. If $|L/\lambda_2(L)| = p^d$, then d is the minimal generator number of L. Similar to the p-group case, one can observe that this is indeed the cardinality of a generating set of smallest possible size.

Each Lie p-ring L has a central series $L = L_1 \geq \ldots \geq L_n \geq \{0\}$ with quotients of order p. Choose $l_i \in L_i \setminus L_{i+1}$ for $1 \leq i \leq n$. Then $(l_1, \ldots, l_n)$ is a generating set of L satisfying that $pl_i \in L_{i+1}$ and $l_il_j \in L_{i+1}$ for $1 \leq j < i \leq n$. We call such a generating sequence a basis for L and we say that L has dimension n.

3 LiePRings in GAP

This package introduces a new datastructure that allows to define and compute with Lie p-rings in GAP. We first describe this datastructure in the case of ordinary Lie p-rings; that is, Lie p-rings for a fixed prime p with given structure constants. Then we show how this datastructure can also be used to define so-called 'generic' Lie p-rings; that is, Lie p-rings with indeterminate prime p.

3.1 Ordinary Lie p-rings

Let p be a prime and let L be a Lie p-ring of order p^n. Let $(l_1, \ldots, l_n)$ be a basis for L. Then there exist coefficients $c_{i,j,k} \in \{0, \ldots, p-1\}$ so that the following relations hold in L for $1 \leq i,j \leq n$ with $i \neq j$:

$$l_i \cdot l_j = \sum_{k=i+1}^{n} c_{i,j,k} l_k,$$

$$pl_i = \sum_{k=i+1}^{n} c_{i,i,k} l_k.$$

These structure constants define the Lie p-ring L. As the multiplication in a Lie p-ring is anticommutative, it follows that $c_{i,j,k} = -c_{j,i,k}$ holds for each k and each $i \neq j$. Thus the structure constants $c_{i,j,k}$ for $i \geq j$ are sufficient to define the Lie p-ring L.

This package contains the new datastructure LiePRing that allows to define Lie p-rings via their structure constants $c_{i,j,k}$. To use this datastructure, we first collect all relevant information into a record as follows:

dim
: *the dimension n of L;*

prime
: *the prime p of L;*

tab
: *a list with structure constants $[c_{1,1}, c_{2,1}, c_{2,2}, c_{3,1}, c_{3,2}, c_{3,3}, \ldots]$.*

Each entry $c_{i,j}$ in the list tab is a list $[k_1, c_{i,j,k_1}, k_2, c_{i,j,k_2}, \ldots]$ so that $k_1 < k_2 < \ldots$ and the entries $c_{i,j,k_1}, c_{i,j,k_2}, \ldots$ are the non-zero structure contants in the product $l_i \cdot l_j$. Thus if $l_i \cdot l_j = 0$, then $c_{i,j}$ is the empty list. If an entry in the list tab is not bound, then it is assumed to be the empty list.

1 ► `LiePRingBySCTable( SC )`
► `LiePRingBySCTableNC( SC )`

These functions create a LiePRing from the structure constants table record SC. The first version checks that the multiplication defined by tab is alternating and satisfies the Jacobi-identity, the second version assumes that this is the case and omits these checks. These checks can also be carried out independently via the following function.

2 ► `CheckIsLiePRing( L )`

This function takes as input an object L created via LiePRingBySCTableNC and checks that the Jacobi identity holds in this ring.

The following example creates the Lie 2*-ring of order* 8 *with trivial multiplication.*

```
gap> SC := rec( dim := 3, prime := 2, tab := [] );;
gap> L := LiePRingBySCTable(SC);
<LiePRing of dimension 3 over prime 2>
gap> l := BasisOfLiePRing(L);
[ l1, l2, l3 ]
gap> l[1]*l[2];
0
gap> 2*l[1];
0
gap> l[1] + l[2];
l1 + l2
```

The next example creates a LiePRing of order 5^4 *with non-trivial multiplication.*

```
gap> SC := rec( dim := 4, prime := 5, tab := [ [], [3, 1], [], [4, 1]]);;
gap> L := LiePRingBySCTableNC(SC);;
gap> ViewPCPresentation(L);
[l2,l1] = l3
[l3,l1] = l4
```

3.2 Generic Lie p-rings

In a generic Lie p-ring, p is allowed to be an indeterminate and the structure constants are allowed to be rational functions over a polynomial ring in a finite set of commuting indeterminates. It is generally assumed that the indeterminate with name p represents the prime, the indeterminate with name w represents the smallest primitive root modulo the prime and there are further predefined indeterminates with the names x, y, z, t, j, k, m, n, r, s, u and v. These indeterminates are used in the database of Lie p-rings and they can be obtained via

1 ► `IndeterminateByName( string )`

The structure constants records for generic Lie p-rings are similar to those for ordinary Lie p-rings, but have the additional entry param which is a list containing all indeterminates used in the considered Lie p-ring. We exhibit an example.

```
gap> p := IndeterminateByName("p");;
gap> x := IndeterminateByName("x");;
gap> S := rec( dim := 5,
>              param := [ x ],
>              prime := p,
>              tab := [ [ 4, 1 ], [ 3, 1 ], [ 5, x ], [ 4, 1 ], [ 5, 1 ] ] );;
gap> L := LiePRingBySCTable(S);
<LiePRing of dimension 5 over prime p with parameters [ x ]>
gap> ViewPCPresentation(L);
p*l1 = l4
p*l2 = x*l5
[l2,l1] = l3
[l3,l1] = l4
[l3,l2] = l5
gap> l := BasisOfLiePRing(L);
[ l1, l2, l3, l4, l5 ]
gap> p*l[1];
l4
```

```
gap> l[1]+l[2];
l1 + l2
gap> l[1]*l[2];
-1*l3
```

3.3 Specialising Lie p-rings

A generic Lie p-ring defines a family of ordinary Lie p-rings by evaluating the parameters contained in its presentation. It is generally assumed that the indeterminate p is evaluated to a rational prime P and the indeterminate w is evaluated to the smallest primitive root modulo P (this can be determined via PrimitiveRootMod(P)). All other indeterminates can take arbitrary integer values (usually these values are in $\{0, \ldots, P-1\}$, but other choices are possible as well). The following functions allow to evaluate the indeterminates.

1 ► `SpecialiseLiePRing(L, P, para, vals)`

takes as input a generic Lie p-ring L, a rational prime P, a list of indeterminates para and a corresponding list of values vals. The function returns a new Lie p-ring in which the prime p is evaluated to P, the parameter w is evaluated to PrimitiveRootMod(P) and the parameters in para are evaluated to vals.

2 ► `SpecialisePrimeOfLiePRing(L, P)`

this is a shortcut for SpecialiseLiePRing(L, P, [], []). We exhibit a some example applications.

```
gap> p := IndeterminateByName("p");;
gap> w := IndeterminateByName("w");;
gap> x := IndeterminateByName("x");;
gap> y := IndeterminateByName("y");;
gap> S := rec( dim := 7,
>              param := [ w, x, y ],
>              prime := p,
>              tab := [ [  ], [ 6, 1 ], [ 6, 1 ], [ 7, 1 ], [  ],
>                       [ 6, x, 7, y ], [  ], [ 7, 1 ], [ 6, w ] ] );;
gap> L := LiePRingBySCTable(S);
<LiePRing of dimension 7 over prime p with parameters [ w, x, y ]>
gap> ViewPCPresentation(L);
p*l2 = l6
p*l3 = x*l6 + y*l7
[l2,l1] = l6
[l3,l1] = l7
[l4,l2] = l7
[l4,l3] = w*l6
gap> SpecialiseLiePRing(L, 7, [x, y], [0,0]);
<LiePRing of dimension 7 over prime 7>
gap> ViewPCPresentation(last);
7*l2 = l6
[l2,l1] = l6
[l3,l1] = l7
[l4,l2] = l7
[l4,l3] = 3*l6
gap> SpecialiseLiePRing(L, 11, [x, y], [0,10]);
<LiePRing of dimension 7 over prime 11>
gap> ViewPCPresentation(last);
11*l2 = l6
11*l3 = 10*l7
```

```
[l2,l1] = l6
[l3,l1] = l7
[l4,l2] = l7
[l4,l3] = 2*l6
gap> Cartesian([0,1],[0,1]);
[ [ 0, 0 ], [ 0, 1 ], [ 1, 0 ], [ 1, 1 ] ]
gap> List(last, v -> SpecialiseLiePRing(L, 2, [x,y], v));
[ <LiePRing of dimension 7 over prime 2>,
  <LiePRing of dimension 7 over prime 2>,
  <LiePRing of dimension 7 over prime 2>,
  <LiePRing of dimension 7 over prime 2> ]
```

It is not necessary to specialise all parameters at once. In particular, it is possible to leave the prime p as indeterminate and specialize only some of the parameters. (Except for w which is linked to p.)

```
gap> SpecialiseLiePRing(L, p, [x], [0]);
<LiePRing of dimension 7 over prime p with parameters [ y, w ]>
gap> ViewPCPresentation(last);
p*l2 = l6
p*l3 = y*l7
[l2,l1] = l6
[l3,l1] = l7
[l4,l2] = l7
[l4,l3] = w*l6
gap> SpecialiseLiePRing(L, p, [y], [3]);
<LiePRing of dimension 7 over prime p with parameters [ x, w ]>
gap> ViewPCPresentation(last);
p*l2 = l6
p*l3 = x*l6 + 3*l7
[l2,l1] = l6
[l3,l1] = l7
[l4,l2] = l7
[l4,l3] = w*l6
```

It is also possible to specialise the prime only, but leave all or some of the parameters indeterminate. Note that specialising p also specialises w. Again, we continue to use the generic Lie p-ring L as above.

```
gap> SpecialisePrimeOfLiePRing(L, 29);
<LiePRing of dimension 7 over prime 29 with parameters [ y, x ]>
gap> ViewPCPresentation(last);
29*l2 = l6
29*l3 = x*l6 + y*l7
[l2,l1] = l6
[l3,l1] = l7
[l4,l2] = l7
[l4,l3] = 2*l6
```

3 ► `LiePValues(K)`

if K is obtained by specialising, then this attribute is set and contains the parameters that have been specialised and their values.

```
gap>  L := LiePRingsByLibrary(6)[14];
<LiePRing of dimension 6 over prime p with parameters [ x ]>
gap>  K := SpecialisePrimeOfLiePRing(L, 5);
<LiePRing of dimension 6 over prime 5 with parameters [ x ]>
gap> LiePValues(K);
[ [ p, w ], [ 5, 2 ] ]
```

3.4 Subrings of Lie p-rings

Let L be a Lie p-ring with basis $(l_1, \ldots, l_n)$ and let U be a subring of L. Then U is a Lie p-ring and thus also has a basis $(u_1, \ldots, u_m)$. For $1 \leq i \leq m$ we define the coefficients $a_{i,j} \in \{0, \ldots, p-1\}$ via

$$u_i = \sum_{j=1}^{n} a_{i,j} l_i$$

and we denote with A the matrix with entries $a_{i,j}$. We say that the basis $(u_1, \ldots, u_m)$ is induced if A is in upper triangular form. Further, the basis $(u_1, \ldots, u_m)$ is canonical if A is in upper echelon form; that is, it is upper triangular, each row in A has leading entry 1 *and there are* 0*'s above the leading entry. Note that a canonical basis is unique for the subring.*

1 ► `LiePSubring(L, gens)`

Let L be a (generic or ordinary) Lie p-ring and let gens be a set of elements in L. This function determines a canonical basis for the subring generated by gens in L and returns the LiePSubring of L generated by gens. Note that this function may have strange effects for generic Lie p-rings as the following example shows.

```
gap> L := LiePRingsByLibrary(6)[100];
<LiePRing of dimension 6 over prime p>
gap> l := BasisOfLiePRing(L);
[ l1, l2, l3, l4, l5, l6 ]
gap> U := LiePSubring(L, [5*l[1]]);
<LiePRing of dimension 3 over prime p>
gap> BasisOfLiePRing(U);
[ l1, l4, l6 ]
gap>  K := SpecialisePrimeOfLiePRing(L, 5);
<LiePRing of dimension 6 over prime 5>
gap>  b := BasisOfLiePRing(K);
[ l1, l2, l3, l4, l5, l6 ]
gap> LiePSubring(K, [5*b[1]]);
<LiePRing of dimension 2 over prime 5>
gap>  BasisOfLiePRing(last);
[ l4, l6 ]
gap> K := SpecialisePrimeOfLiePRing(L, 7);
<LiePRing of dimension 6 over prime 7>
gap> b := BasisOfLiePRing(K);
[ l1, l2, l3, l4, l5, l6 ]
gap> U := LiePSubring(K, [5*b[1]]);
<LiePRing of dimension 3 over prime 7>
gap> BasisOfLiePRing(U);
[ l1, l4, l6 ]
```

2 ► `LiePIdeal(L, gens)`

return the ideal of L generated by gens. This function computes a an induced basis for the ideal.

```
gap> LiePIdeal(L, [l[1]]);
<LiePRing of dimension 5 over prime p>
gap> BasisOfLiePRing(last);
[ l1, l3, l4, l5, l6 ]
```

3 ► `LiePQuotient(L, U)`

return a Lie p-ring isomorphic to L/U where U must be an ideal of L. This function requires that L is an ordinary Lie p-ring.

```
gap> LiePIdeal(K, [b[1]]);
<LiePRing of dimension 5 over prime 7>
gap> LiePIdeal(K, [b[2]]);
<LiePRing of dimension 4 over prime 7>
gap> LiePQuotient(K,last);
<LiePRing of dimension 2 over prime 7>
```

3.5 Elementary functions

The functions described in this section work for ordinary and generic Lie p-rings and their subrings.

1 ► `PrimeOfLiePRing(L)`

returns the underlying prime. This can either be an integer or an indeterminate.

2 ► `BasisOfLiePRing(L)`

returns a basis for L.

3 ► `DimensionOfLiePRing(L)`

returns the dimension of L.

4 ► `ParametersOfLiePRing(L)`

returns the list of indeterminates involved in L. If L is a subring of a Lie p-ring defined by structure constants, then the parameters of the parent are returned.

5 ► `ViewPCPresentation(L)`

prints the presentation for L with respect to its basis.

3.6 Series of subrings

Let L be a generic or ordinary Lie p-ring or a subring of such such a Lie p-ring.

1 ► `LiePLowerCentralSeries(L)`

returns the lower central series of L.

2 ► `LiePLowerPCentralSeries(L)`

returns the lower exponent-p central series of L.

3 ► `LiePDerivedSeries(L)`

returns the derived series of L.

4 ► `LiePMinimalGeneratingSet(L)`

returns a minimal generating set of L; that is, a generating set of smallest possible size.

3.7 The Lazard correspondence

The following function has been implemented by Willem de Graaf. It uses the Baker-Campbell-Hausdorff formula as described in [CdGVL12] and it is based on the Liering package [CdG10].

1 ► `PGroupByLiePRing(L)`

Let L be an ordinary Lie p-ring with $cl(L) < p$. *Then this function returns the p-group G obtained from L via the Lazard correspondence.*

4 The Database

This package gives access to the database of Lie p-rings of order at most p^7 as determined by Mike Newman, Eamonn O'Brien and Michael Vaughan-Lee, see [NOVL03] and [OVL05]. A description of the database can also be found in [VL13].

For each $n \in \{1,\dots,7\}$ this package contains a (finite) list of generic presentations of Lie p-rings. For each prime $p \geq 5$, each of the generic Lie p-rings gives rise to a family of Lie p-rings over the considered prime p by specialising the indeterminates to a certain list of values. The resulting lists of Lie p-rings provides a complete and irredundant set of isomorphism type representatives of the Lie p-rings of order p^n. The generic Lie p-rings of p-class at most 2 can also be considered for the prime $p = 3$ and yield a list of isomorphism type representatives for the Lie p-rings of order 3^n and p-class at most 2.

The Lazard correspondence has been used to check the correctness of the database of Lie p-rings: for various small primes it has been checked that the Lie p-rings of this database define non-isomorphic finite p-groups.

In the following we describe functions to access the database. Throughout this chapter, we assume that dim $\in \{1,\dots,7\}$ and P is a prime with $P \neq 2$.

4.1 Accessing Lie p-rings

1 ► `LiePRingsByLibrary( dim )`
► `LiePRingsByLibrary( dim, gen, cl )`

returns the generic Lie p-rings of dimension dim in the database. The second form returns the Lie p-rings of minimal generator number gen and p-class cl only.

2 ► `LiePRingsByLibrary( dim, P )`
► `LiePRingsByLibrary( dim, P, gen, cl )`

returns isomorphism type representatives of ordinary Lie p-rings of dimension dim for the prime P. The second form returns the Lie p-rings of minimal generator number gen and p-class cl only. The function assumes $P \geq 3$ and for $P = 3$ there are only the Lie p-rings of p-class at most 2 available.

The first example yields the generic Lie p-rings of dimension 4.

```
gap> LiePRingsByLibrary(4);
[ <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
```

```
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p>,
  <LiePRing of dimension 4 over prime p> ]
```

The next example yields the isomorphism type representatives of Lie p-rings of dimension 3 *for the prime* 5.

```
gap> LiePRingsByLibrary(3, 5);
[ <LiePRing of dimension 3 over prime 5>,
  <LiePRing of dimension 3 over prime 5>,
  <LiePRing of dimension 3 over prime 5>,
  <LiePRing of dimension 3 over prime 5>,
  <LiePRing of dimension 3 over prime 5> ]
```

The following example extracts the generic Lie p-rings of dimension 5 *with minimal generator number* 2 *and p-class* 4.

```
gap> LiePRingsByLibrary(5, 2, 4);
[ <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p> ]
```

Finally, we determine the isomorphism type representatives of Lie p-rings of dimension 5*, minimal generator number* 2 *and p-class* 4 *for the prime* 7.

```
gap> LiePRingsByLibrary(5, 7, 2, 4);
[ <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7>,
  <LiePRing of dimension 5 over prime 7> ]
```

4.2 Numbers of Lie p-rings

1 ► `NumberOfLiePRings( dim )`

returns the number of generic Lie p-rings in the database of the considered dimension for dim $\{1,\dots,7\}$.

```
gap> List([1..7], x -> NumberOfLiePRings(x));
[ 1, 2, 5, 15, 75, 542, 4773 ]
```

2 ► `NumberOfLiePRings( dim, P )`

returns the number of isomorphism types of ordinary Lie p-rings of order P^{dim} *in the database. If* $P \geq 5$*, then this is the number of all isomorphism types of Lie p-rings of order* P^{dim} *and if* $P = 3$ *then this is the number of all isomorphism types of Lie p-rings of p-class at most* 2*. If* $P \geq 7$*, then this number coincides with NumberSmallGroups(*P^{dim}*).*

3 ► `NumberOfLiePRingsInFamily( L )`

returns the number of Lie p-rings associated to L as a polynomial in p and possibly some residue classes.

```
gap> L := LiePRingsByLibrary(7)[780];
<LiePRing of dimension 7 over prime p with parameters
[ x, y, z, t, s, u, v ]>
gap> NumberOfLiePRingsInFamily(L);
-1/3*p^5*(p-1,3)+p^5-1/3*p^4*(p-1,3)+p^4-1/3*p^3*(p-1,3)+p^3-1/3*p^2*(p-1,3)
+p^2-p*(p-1,3)+3*p-3/2*(p-1,3)+9/2
```

4.3 Searching the database

We now consider a generic Lie p-ring L from the database and consider the family of ordinary Lie p-rings that arise from it.

1 ► `LiePRingsInFamily( L, P )`

takes as input a generic Lie p-ring L from the database and a prime P and returns all Lie p-rings determined by L and P up to isomorphism. This function returns fail if the generic Lie p-ring does not exist for the special prime P; this may be due to the conditions on the prime or (if $P = 3$*) to the p-class of the Lie p-ring.*

```
gap> L := LiePRingsByLibrary(7)[118];
<LiePRing of dimension 7 over prime p with parameters [ x, y ]>
gap> LibraryConditions(L);
[ "[x,y]~[x,-y]", "p=1 mod 4" ]
gap> LiePRingsInFamily(L, 7);
fail
gap> Length(LiePRingsInFamily(L,13));
91
gap> 13^2;
169
```

The following example shows how to determine all Lie p-rings of dimension 5 *and p-class* 4 *over the prime* 29 *up to isomorphism.*

```
gap> L := LiePRingsByLibrary(5);;
gap> L := Filtered(L, x -> PClassOfLiePRing(x)=4);
[ <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p>,
  <LiePRing of dimension 5 over prime p> ]
gap> K := List(L, x-> LiePRingsInFamily(x, 29));
[ [ <LiePRing of dimension 5 over prime 29> ],
  [ <LiePRing of dimension 5 over prime 29> ],
  [ <LiePRing of dimension 5 over prime 29> ], fail, fail,
  [ <LiePRing of dimension 5 over prime 29> ],
  [ <LiePRing of dimension 5 over prime 29> ],
  [ <LiePRing of dimension 5 over prime 29> ],
  [ <LiePRing of dimension 5 over prime 29> ],
  [ <LiePRing of dimension 5 over prime 29> ],
  [ <LiePRing of dimension 5 over prime 29> ], fail, fail,
  [ <LiePRing of dimension 5 over prime 29> ],
  [ <LiePRing of dimension 5 over prime 29> ] ]
gap> K := Filtered(Flat(K), x -> x<>fail);
[ <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29>,
  <LiePRing of dimension 5 over prime 29> ]
```

4.4 More details

Let L be a Lie p-ring from the database. Then the following additional attributes are available.

1 ► `LibraryName(L)`

returns a string with the name of L in the database. See p567.pdf for further background.

2 ► `ShortPresentation(L)`

returns a string exhibiting a short presentation of L.

3 ► `LibraryConditions(L)`

returns the conditions on L. This is a list of two strings. The first string exhibits the conditions on the parameters of L, the second shows the conditions on primes.

4 ► `MinimalGeneratorNumberOfLiePRing(L)`

returns the minimial generator number of L.

5 ► `PClassOfLiePRing(L)`

returns the p-class of L.

```
gap> L := LiePRingsByLibrary(7)[118];
<LiePRing of dimension 7 over prime p with parameters [ x, y ]>
gap> LibraryName(L);
"7.118"
gap> LibraryConditions(L);
[ "[x,y]~[x,-y]", "p=1 mod 4" ]
```

All of the information listed in this section is inherited when L is specialised.

```
gap> L := LiePRingsByLibrary(7)[118];
<LiePRing of dimension 7 over prime p with parameters [ x, y ]>
gap> K := SpecialiseLiePRing(L, 13, ParametersOfLiePRing(L), [0,0]);
<LiePRing of dimension 7 over prime 13>
gap> LibraryName(K);
"7.118"
gap> LibraryConditions(K);
[ "[x,y]~[x,-y]", "p=1 mod 4" ]
```

The following example shows how to find a Lie p-ring with a given name in the database.

```
gap> L := LiePRingsByLibrary(7);;
gap> Filtered(L, x -> LibraryName(x) = "7.1010")[1];
<LiePRing of dimension 7 over prime p>
```

4.5 Special functions for dimension 7

The database of Lie p-rings of dimension 7 is very large and it may be time-consuming (or even impossible due to storage problems) to generate all Lie p-rings of dimension 7 for a given prime P.

Thus there are some special functions available that can be used to access a particular set of Lie p-rings of dimension 7 only. In particular, it is possible to consider the descendants of a single Lie p-ring of smaller dimension by itself. The Lie p-rings of this type are all stored in one file of the library. Thus, equivalently, it is possible to access the Lie p-rings in one single file only.

The table LIE_TABLE contains a list of all possible files together with the number of Lie p-rings generated by their corresponding Lie p-rings.

1 ► `LiePRingsDim7ByFile( nr )`

returns the generic Lie p-rings in file number nr.

2 ► `LiePRingsDim7ByFile( nr, P )`

returns the isomorphism types of Lie p-rings in file number nr for the prime P.

```
gap> LIE_TABLE[100];
[ "3gen/gapdec6.139", 1/2*p+(p-1,3)+3/2 ]
gap> LiePRingsDim7ByFile(100);
[ <LiePRing of dimension 7 over prime p>,
  <LiePRing of dimension 7 over prime p>,
  <LiePRing of dimension 7 over prime p>,
  <LiePRing of dimension 7 over prime p>,
  <LiePRing of dimension 7 over prime p with parameters [ x ]> ]
gap> LiePRingsDim7ByFile(100, 7);
[ <LiePRing of dimension 7 over prime 7>,
  <LiePRing of dimension 7 over prime 7>,
  <LiePRing of dimension 7 over prime 7>,
  <LiePRing of dimension 7 over prime 7>,
  <LiePRing of dimension 7 over prime 7>,
  <LiePRing of dimension 7 over prime 7>,
  <LiePRing of dimension 7 over prime 7>,
  <LiePRing of dimension 7 over prime 7> ]
```

4.6 Dimension 8 and maximal class

Recently, Lee and Vaughan-Lee [LVL22] determined the Lie p-rings of dimension 8 with maximal class up to isomorphism. This classification is now also available in the Lie p-ring package via the following functions.

1 ► `LiePRingsByLibraryMC8()`

returns a list of 69 *generic Lie p-rings. For each of these the following function returns the isomorphism types of Lie p-rings in the family for a fixed prime P with $P \geq 5$.*

2 ► `LiePRingsInFamilyMC8(L, P)`

5 Advanced functions for Lie p-rings

This chapter described a few more advanced functions available for generic Lie p-rings.

5.1 Schur multipliers

The package contains a method to determine the Schur multipliers of the Lie p-rings in the family defined by a generic Lie p-ring.

1 ► `LiePSchurMult( L )`

The function takes as input a generic Lie p-ring and determines a list of possible Schur multipliers, each described by its abelian invariants, for the Lie p-rings in the family described by L. For each entry in the list of Schur multipliers there is a description of those parameters which give the considered entry. This description consists of two lists 'units' and 'zeros'. Both consist of rational functions over the parameters of the Lie p-ring. The parameters described by these lists are which evaluate to zero for each rational function in 'zeros' and evaluate not to zero for each rational function in 'units'.

```
gap> LL := LiePRingsByLibrary(7);;
gap> L := Filtered(LL, x -> Length(ParametersOfLiePRing(x))=2)[1];
<LiePRing of dimension 7 over prime p with parameters [ x, y ]>
gap> NumberOfLiePRingsInFamily(L);
p^2-p
gap> RingInvariants(L);
rec( units := [ x ], zeros := [  ] )
gap> ss := LiePSchurMult(L);
[ rec( norm := [ p ], units := [ x, y ], zeros := [ x*y^2-x*y+1 ] ),
  rec( norm := [ p^2 ], units := [ x ], zeros := [ x*y ] ),
  rec( norm := [ p ], units := [ x, x*y^2-x*y+1, y ], zeros := [  ] ) ]
```

In this example, L defines a generic Lie p-rings with two parameters and the RingInvariants of L show that the parameter x should be non-zero. The function LiePSchurMult(L) yields that there are two possible Schur multipliers for the Lie p-rings in the family defined by L: the cyclic groups of order p and of order p^2. *The second option only arises if* $xy = 0$ *and thus, as x is non-zero, if* $y = 0$.

The package also contains a function that tries to determine the numbers of values of the parameters satisfying the conditions of a description of a Schur multiplier. This succeeds in many cases and returns a polynomial in p in this case. If it does not succeed then it returns fail.

2 ► `ElementNumbers( pp, ss )`

We continue the above example.

```
gap> ElementNumbers(ParametersOfLiePRing(L), ss);
rec( norms := [ [ p^2 ], [ p ] ], numbs := [ p-1, p^2-2*p+1 ] )
```

5.2 Automorphism groups

The package contains a function that determines a description for the automorphism groups of the Lie p-rings in the family defined by a generic Lie p-ring.

1 ► `AutGrpDescription( L )`

Each automorphism of L is defined by its images on a generating set of L. If $l_1, \cdots, l_n$ is a basis of L and $l_1, \cdot\cdot, l_d$ is a generating set, then each automorphism is defined by the images of $l_1, \cdot\cdot, l_d$ and each image is an integral linear combination of the basis elements $l_1, \cdot\cdot, l_n$. The function AutGrpDescription returns a matrix containing a description of the coefficients in each linear combination and a list of relations among these coefficients. We consider two examples.

```
gap> L := Filtered(LL, x -> Length(ParametersOfLiePRing(x))=2)[1];
<LiePRing of dimension 7 over prime p with parameters [ x, y ]>
gap> AutGroupDescription(L);
rec( auto := [ [ 1, 0, A13, A14, A15, A16, A17 ],
               [ 0, 1, A23, A24, A25, A26, A27 ] ],
     eqns := [ [  ], [  ] ] )
gap> L := Filtered(LL, x -> Length(ParametersOfLiePRing(x))=2)[2];
<LiePRing of dimension 7 over prime p with parameters [ x, y ]>
gap> AutGroupDescription(L);
rec( auto := [ [ A22^3, 0, A13, A14, A15, A16, A17 ],
               [ 0, A22, A23, A24, A25, A26, A27 ] ],
     eqns := [ [ A22*A24-1/2*A23^2, A22^2*y-y,
                 A22*A23^2*y-2*A24*y, A22^4-1,
                 A23^4*y-4*A24^2*y, A22^3*A23^2-2*A24,
                 A22^2*A23^4-4*A24^2, A22*A23^6-8*A24^3,
                 A23^8-16*A24^4 ] ] )
```

In both cases, L is generated by the first two entries in its basis and hence the automorphism group matrix has two rows and seven columns. In the first case, L has p^{10} automorphisms inducing the identity on the Frattini-quotient of L. In the second case, the automorphism group matrix shows that each automorphism induces a certain type of diagonal matrix on the Frattini-quotient of L and there are further equations among the coefficients of the matrix. These further equations are equivalent to $A22^2 = 1$ and $A24 = A22A23^2/2$. Hence L has $2p^9$ automorphisms.

The entry eqns is a list of lists. The equations in the ith entry of this list have to be satisfied mod p^i.

In a few special cases, the function returns a list of possible automorphisms together with related equations and conditions. We exhibit an example.

```
gap> L := LiePRingsByLibrary(7)[489];
<LiePRing of dimension 7 over prime p with parameters [ x ]>
gap> AutGroupDescription(L);
[ rec( auto := [ [ 1, 0, A13, A14, A15, A16, A17 ],
                 [ 0, 1, A23, A24, A25, A26, A27 ] ],
       comment := "p^8 automorphisms",
       eqns := [ [ A13^2*x-A13*A23+2*A15*x+A14-A25,
              -A13*A23*x+A14*x+A23^2-A25*x-2*A24 ] ] ),
  rec( auto := [ [ 0, A12, A13, A14, A15, A16, A17 ],
                 [ -x, 0, A23, A24, A25, A26, A27 ] ],
      comment := "p^8 automorphisms when x <> 0 mod p",
      eqns := [ [ A12^2*A24*x-A12*A13*A23*x+A12*A13*x^2
                  +2*A12*A15*x^2+A12*A14*x-A13^2*x+A13*x+A15*x-A14,
                  -A12^2*A23*x^3+A12*A13*x^3+A12*A23^2*x-A12*A25*x^2
                  -2*A12*A24*x+A13*A23*x+A13*x^2-A15*x^2+A23*x+A25*x-A24 ],
                [ A12*x+1 ] ] ) ]
```

In this example $A12x = -1$ *modulo* p^2*. We note that different choices for* $A12$ *do not give different automorphisms. Hence a single solution for* $A12$ *is sufficient to describe all automorphisms.*

Bibliography

[CdG10] *Serena Cicalò and Willem A. de Graaf.* Liering*, 2010. A GAP 4 package.*

[CdGVL12] *Serena Cicalò, Willem A. de Graaf, and Michael R. Vaughan-Lee. An effective version of the Lazard correspondence.* J. Algebra*, 352:430–450, 2012.*

[Khu98] *E. I. Khukhro.* p-automorphisms of finite p-groups*, volume 246 of* London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.*

[LVL22] *Seungjai Lee and Michael Vaughan-Lee. The groups and nilpotent lie rings of order p^8 and maximal class.* To appear*, 2022.*

[NOVL03] *Mike F. Newman, Eamonn A. O'Brien, and Michael R. Vaughan-Lee. Groups and nilpotent Lie rings whose order is the sixth power of a prime.* J. Alg.*, 278:383 – 401, 2003.*

[OVL05] *Eamonn A. O'Brien and Michael R. Vaughan-Lee. The groups with order p^7 for odd prime p.* J. Algebra*, 292(1):243–258, 2005.*

[VL13] *Michael R. Vaughan-Lee. The nilpotent Lie rings of order p^k for $k \leq 7$, 2013. See within this package under /lib/notes/p567.pdf.*

Index

This index covers only this manual. A page number in italics refers to a whole section which is devoted to the indexed subject. Keywords are sorted with case and spaces ignored, e.g., "PermutationCharacter" comes before "permutation group".