

PerfGrp

GAP Library of Finite Perfect Groups

1.0.0

11 August 2026

Alexander Hulpke
Derek F. Holt
Wilhelm Plesken
Volkmar Felsch

Alexander Hulpke Email: hulpke@math.colostate.edu
Homepage: <https://www.math.colostate.edu/~hulpke/>
Address: Department of Mathematics
Colorado State University
1874 Campus Delivery
Fort Collins, CO 80523-1874
USA

Derek F. Holt Email: D.F.Holt@warwick.ac.uk
Homepage: <https://homepages.warwick.ac.uk/~mareg/>

Abstract

The PerfGrp package provides the GAP library of finite perfect groups, and functionality for determining the perfect subgroups of a finite group.

Copyright

© 1989–2026 by the authors.

This package is free software; you can redistribute it and/or modify it under the terms of the [GNU General Public License](#) as published by the Free Software Foundation; either version 2 of the License, or (at your option) any later version.

Contents

1	The Perfect Groups Library	4
1.1	Accessing the Perfect Groups Library	4
1.2	Data Not Distributed with the Package	8
1.3	More about the Perfect Groups Library	8
	References	11
	Index	12

Chapter 1

The Perfect Groups Library

The **GAP** library of finite perfect groups provides, up to isomorphism, a list of all perfect groups whose sizes are less than $2 \cdot 10^6$. The groups of orders up to 10^6 have been enumerated by Derek F. Holt and Wilhelm Plesken and published in their book “Perfect Groups” [HP89]. For orders $n = 86016$, 368640 , or 737280 this work only counted the groups (but did not explicitly list them), the groups of orders $n = 61440$, 122880 , 172032 , 245760 , 344064 , 491520 , 688128 , or 983040 were omitted.

We are grateful to Derek Holt and Wilhelm Plesken for making their groups available to the **GAP** community by contributing their files. It should be noted that their book contains a lot of further information for many of the library groups. So we would like to recommend it to any **GAP** user who is interested in the groups. The library of these has been brought into **GAP** format by Volkmar Felsch.

Several additional groups omitted from the book “Perfect Groups” have also been included. Two groups -- one of order 450000 with a factor group of type A_6 and the one of order 962280 -- were found by Jack Schmidt in 2005. Two groups of order 243000 and one each of orders 729000 , 871200 , 878460 were found in 2020 by Alexander Hulpke.

The perfect groups of size less than $2 \cdot 10^6$ which had not been classified in the work of Holt and Plesken have been enumerated by Alexander Hulpke; this classification is described in [Hul22], which also gives an account of the methods used and of the verification of the resulting data. These groups are stored directly and provide less construction information in their names.

As all groups are stored by presentations, a permutation representation is obtained by coset enumeration. Note that some of the library groups do not have a faithful permutation representation of small degree. Computations in these groups may be rather time consuming.

The perfect subgroups of a given finite group can be determined with `RepresentativesPerfectSubgroups` (**Reference:** `RepresentativesPerfectSubgroups`) and `ConjugacyClassesPerfectSubgroups` (**Reference:** `ConjugacyClassesPerfectSubgroups`), which are declared by the **GAP** library but require this package in order to work.

1.1 Accessing the Perfect Groups Library

1.1.1 `SizesPerfectGroups`

▷ `SizesPerfectGroups()` (function)

This is the ordered list of all numbers up to $2 \cdot 10^6$ that occur as sizes of perfect groups. One can iterate over part of the perfect groups library with:

Example

```
gap> for n in Intersection([100..500],SizesPerfectGroups()) do
>   for k in [1..NrPerfectGroups(n)] do
>     pg := PerfectGroup(n,k);
>     od;
>   od;
```

1.1.2 PerfectGroup

- ▷ `PerfectGroup([filt, ]size[, n])` (function)
- ▷ `PerfectGroup([filt, ]sizenumberpair)` (function)

returns a group which is isomorphic to the library group specified by the size number [*size*, *n*] or by the two separate arguments *size* and *n*, assuming a default value of $n = 1$. The optional argument *filt* defines the filter in which the group is returned. Possible filters so far are `IsPermGroup` (**Reference: IsPermGroup**) and `IsSubgroupFpGroup` (**Reference: IsSubgroupFpGroup**). In the latter case, the generators and relators used coincide with those given in [HP89]. The default filter is `IsPermGroup` (**Reference: IsPermGroup**).

The data for the orders 1376256 and 1966080 is not distributed with this package; for these, this function enters a break loop explaining how to obtain the missing file, see 1.2.

Example

```
gap> G := PerfectGroup(IsPermGroup,6048,1);
U3(3)
gap> G:=PerfectGroup(IsPermGroup,823080,2);
A5 2^1 19^2 C 19^1
gap> NrMovedPoints(G);
6859
gap> G:=PerfectGroup(1866240,12);
PG1866240.12
gap> NrMovedPoints(G);
270
```

1.1.3 PerfectIdentification

- ▷ `PerfectIdentification(G)` (attribute)

This attribute is set for all groups obtained from the perfect groups library and has the value [*size*, *nr*] if the group is obtained with these parameters from the library.

1.1.4 NumberPerfectGroups

- ▷ `NumberPerfectGroups(size)` (function)
- ▷ `NrPerfectGroups(size)` (function)
- ▷ `NumberPerfectLibraryGroups(size)` (function)
- ▷ `NrPerfectLibraryGroups(size)` (function)

returns the number of non-isomorphic perfect groups of size *size* for each positive integer *size* up to $2 \cdot 10^6$. Additionally, for odd *size* an answer is returned (odd order groups are

solvable). For any other argument out of range it returns fail. *NrPerfectGroups* is a synonym for *NumberPerfectGroups*. Moreover *NumberPerfectLibraryGroups* (and its synonym *NrPerfectLibraryGroups*) exist for historical reasons, and return 0 instead of fail for arguments outside the library scope.

1.1.5 SizeNumbersPerfectGroups

▷ `SizeNumbersPerfectGroups(factor1, factor2, ...)` (function)

`SizeNumbersPerfectGroups` returns a list of pairs, each entry consisting of a group order and the number of those groups in the library of perfect groups that contain the specified factors *factor1*, *factor2*, ... among their composition factors.

Each argument must either be the name of a nonabelian simple group or an integer which stands for the product of the sizes of one or more cyclic factors. (In fact, the function replaces all integers among the arguments by their product.)

The following text strings are accepted as simple group names.

- A_n or $A(n)$ for the alternating groups A_n , $5 \leq n \leq 9$, for example A_5 or $A(6)$.
- $L_n(q)$ or $L(n, q)$ for $PSL(n, q)$, where $n \in \{2, 3\}$ and q a prime power, ranging
 - for $n = 2$ from 4 to 125
 - for $n = 3$ from 2 to 5
- $U_n(q)$ or $U(n, q)$ for $PSU(n, q)$, where $n \in \{3, 4\}$ and q a prime power, ranging
 - for $n = 3$ from 3 to 5
 - for $n = 4$ from 2 to 2
- $Sp_4(4)$ or $S(4, 4)$ for the symplectic group $Sp(4, 4)$,
- $Sz(8)$ for the Suzuki group $Sz(8)$,
- M_n or $M(n)$ for the Mathieu groups M_{11} , M_{12} , and M_{22} , and
- J_n or $J(n)$ for the Janko groups J_1 and J_2 .

Note that, for most of the groups, the preceding list offers two different names in order to be consistent with the notation used in [HP89] as well as with the notation used in the `DisplayCompositionSeries` (**Reference:** `DisplayCompositionSeries`) command of **GAP**. However, as the names are compared as text strings, you are restricted to the above choice. Even expressions like $L_2(2^5)$ are not accepted.

As the use of the term $PSU(n, q)$ is not unique in the literature, we mention that in this library it denotes the factor group of $SU(n, q)$ by its centre, where $SU(n, q)$ is the group of all $n \times n$ unitary matrices with entries in $GF(q^2)$ and determinant 1.

The purpose of the function is to provide a simple way to formulate a loop over all library groups which contain certain composition factors.

As this function runs over the whole library, it requires the data files that are not distributed with this package, whatever its arguments are, see 1.2.

1.1.6 DisplayInformationPerfectGroups

- ▷ `DisplayInformationPerfectGroups(size[, n])` (function)
 ▷ `DisplayInformationPerfectGroups(sizenumberpair)` (function)

`DisplayInformationPerfectGroups` displays some invariants of the n -th group of order `size` from the perfect groups library.

If no value of n has been specified, the invariants will be displayed for all groups of size `size` available in the library.

Alternatively, also a list of length two may be entered as the only argument, with entries `size` and n .

The information provided for G includes the following items:

- a headline containing the size number [`size`, n] of G in the form `size.n` (the suffix `.n` will be suppressed if, up to isomorphism, G is the only perfect group of order `size`),
- a message if G is simple or quasisimple, i.e., if the factor group of G by its centre is simple,
- the “description” of the structure of G as it is given by Holt and Plesken in [HP89] (see below),
- the size of the centre of G (suppressed, if G is simple),
- the prime decomposition of the size of G ,
- orbit sizes for a faithful permutation representation of G which is provided by the library (see below),
- a reference to each occurrence of G in the tables of section 5.3 of [HP89]. Each of these references consists of a class number and an internal number (i, j) under which G is listed in that class. For some groups, there is more than one reference because these groups belong to more than one of the classes in the book.

Example

```
gap> DisplayInformationPerfectGroups( 30720, 3 );
#I Perfect group 30720:  A5 ( 2^4 E N 2^1 E 2^4 ) A
#I size = 2^11*3*5 orbit size = 240
#I Holt-Plesken class 1 (9,3)
gap> DisplayInformationPerfectGroups( 30720, 6 );
#I Perfect group 30720:  A5 ( 2^4 x 2^4 ) C N 2^1
#I centre = 2 size = 2^11*3*5 orbit size = 384
#I Holt-Plesken class 1 (9,6)
gap> DisplayInformationPerfectGroups( Factorial( 8 ) / 2 );
#I Perfect group 20160.1:  A5 x L3(2) 2^1
#I centre = 2 size = 2^6*3^2*5*7 orbit sizes = 5 + 16
#I Holt-Plesken class 31 (1,1) (occurs also in class 32)
#I Perfect group 20160.2:  A5 2^1 x L3(2)
#I centre = 2 size = 2^6*3^2*5*7 orbit sizes = 7 + 24
#I Holt-Plesken class 31 (1,2) (occurs also in class 32)
#I Perfect group 20160.3:  ( A5 x L3(2) ) 2^1
#I centre = 2 size = 2^6*3^2*5*7 orbit size = 192
#I Holt-Plesken class 31 (1,3)
#I Perfect group 20160.4:  simple group A8
#I size = 2^6*3^2*5*7 orbit size = 8
```

```
#I Holt-Plesken class 26 (0,1)
#I Perfect group 20160.5: simple group L3(4)
#I size = 2^6*3^2*5*7 orbit size = 21
#I Holt-Plesken class 27 (0,1)
```

1.2 Data Not Distributed with the Package

For reasons of size, the data for two orders is *not* shipped with this package: the 1639 groups of order 1376256 and the 7344 groups of order 1966080. All other orders are covered by the data files in the data subdirectory of the package.

How many groups there are of these two orders is recorded in the index file, which is always present, so `NumberPerfectGroups` (1.1.4) and `SizesPerfectGroups` (1.1.1) work as usual. Any function that has to touch the groups themselves, however, enters a break loop with a message naming the file that is missing and where to get it:

Example

```
gap> G := PerfectGroup(1376256, 1);
Error,

For reasons of size, the perfect groups library for orders >10^6 is
not distributed fully by default. To access the group requested, get
the file perf27.grp from
https://github.com/hulpke/extraperfect
and put it in the 'data' subdirectory of the 'perfgrp' package. Then type

return;

to continue in this GAP session (which will read in the file).
```

To proceed, download the file named in the message from <https://github.com/hulpke/extraperfect>, place it in the data subdirectory of the PerfGrp package, and then type `return;`. The interrupted computation continues, reading in the file that has just been added. The two files in question are `perf27.grp` for order 1376256 and `perf33.grp` for order 1966080.

Besides `PerfectGroup` (1.1.2) and `DisplayInformationPerfectGroups` (1.1.6) called with one of these two orders, this also affects `SizeNumbersPerfectGroups` (1.1.5), which iterates over the whole library and therefore needs *both* files, whatever composition factors are asked for.

1.3 More about the Perfect Groups Library

For any library group G , the library files do not only provide a presentation, but, in addition, a list of one or more subgroups $S_1, \dots, S_r$ of G such that there is a faithful permutation representation of G of degree $\sum_{i=1}^r [G : S_i]$ on the set $\{S_i g \mid 1 \leq i \leq r, g \in G\}$ of the cosets of the S_i . This allows one to construct the groups as permutation groups. The function `DisplayInformationPerfectGroups` (1.1.6) displays only the available degree. The message

Example

```
orbit size = 8
```


in the above example means that the available permutation representation is transitive and of degree 8, whereas the message

Example

orbit sizes = 5 + 16

means that a nontransitive permutation representation is available which acts on two orbits of size 5 and 16 respectively.

The notation used in the “description” of a group is explained in section 5.1.2 of [HP89]. We quote the respective page from there:

Within a class $Q\#p$, an isomorphism type of groups will be denoted by an ordered pair of integers (r, n) , where $r \geq 0$ and $n > 0$. More precisely, the isomorphism types in $Q\#p$ of order $p^r|Q|$ will be denoted by $(r, 1), (r, 2), (r, 3), \dots$. Thus Q will always get the size number $(0, 1)$.

In addition to the symbol (r, n) , the groups in $Q\#p$ will also be given a more descriptive name. The purpose of this is to provide a very rough idea of the structure of the group. The names are derived in the following manner. First of all, the isomorphism classes of irreducible F_pQ -modules M with $|Q| \cdot |M| \leq 10^6$, where F_p is the field of order p , are assigned symbols. These will either be simply p^x , where x is the dimension of the module, or, if there is more than one isomorphism class of irreducible modules having the same dimension, they will be denoted by $p^x, p^{x'}$, etc. The one-dimensional module with trivial Q -action will therefore be denoted by p^1 . These symbols will be listed under the description of Q . The group name consists essentially of a list of the composition factors working from the top of the group downwards; hence it always starts with the name of Q itself. (This convention is the most convenient in our context, but it is different from that adopted in the ATLAS [CCN⁺85], for example, where composition factors are listed in the reverse order. For example, we denote a group isomorphic to $SL(2, 5)$ by $A_5 2^1$ rather than $2.A_5$.)

Some other symbols are used in the name, in order to give some idea of the relationship between these composition factors, and splitting properties. We shall now list these additional symbols.

- $\times$ between two factors denotes a direct product of F_pQ -modules or groups.
- C** (for “commutator”) between two factors means that the second lies in the commutator subgroup of the first. Similarly, a segment of the form $(f_1 \times f_2)Cf_3$ would mean that the factors f_1 and f_2 commute modulo f_3 and f_3 lies in $[f_1, f_2]$.
- A** (for “abelian”) between two factors indicates that the second is in the p th power (but not the commutator subgroup) of the first. “A” may also follow the factors, if bracketed.
- E** (for “elementary abelian”) between two factors indicates that together they generate an elementary abelian group (modulo subsequent factors), but that the resulting F_pQ -module extension does not split.
- N** (for “nonsplit”) before a factor indicates that Q (or possibly its covering group) splits down as far as this factor but not over the factor itself. So “ Qf_1Nf_2 ” means that the normal subgroup f_1f_2 of the group has no complement but, modulo f_2, f_1 , does have a complement.

Brackets have their obvious meaning. Summarizing, we have:

- $\times$ = direct product;
- C** = commutator subgroup;

- A** = abelian;
E = elementary abelian; and
N = nonsplit.

Here are some examples.

- (i) $A_5(2^4 E 2^1 E 2^4)A$ means that the pairs $2^4 E 2^1$ and $2^1 E 2^4$ are both elementary abelian of exponent 4.
- (ii) $A_5(2^4 E 2^1 A)C 2^1$ means that $O_2(G)$ is of symplectic type 2^{1+5} , with Frattini factor group of type $2^4 E 2^1$. The “A” after the 2^1 indicates that G has a central cyclic subgroup $2^1 A 2^1$ of order 4.
- (iii) $L_3(2)((2^1 E) \times (N 2^3 E 2^{3'} A)C) 2^{3'}$ means that the $2^{3'}$ factor at the bottom lies in the commutator subgroup of the pair $2^3 E 2^{3'}$ in the middle, but the lower pair $2^{3'} A 2^{3'}$ is abelian of exponent 4. There is also a submodule $2^1 E 2^{3'}$, and the covering group $L_3(2) 2^1$ of $L_3(2)$ does not split over the 2^3 factor. (Since G is perfect, it goes without saying that the extension $L_3(2) 2^1$ cannot split itself.)

We must stress that this notation does not always succeed in being precise or even unambiguous, and the reader is free to ignore it if it does not seem helpful.

If such a group description has been given in the book for G (and, in fact, this is the case for most of the library groups), it is displayed by `DisplayInformationPerfectGroups` (1.1.6). Otherwise the function provides a less explicit description of the (in these cases unique) Holt–Plesken class to which G belongs, together with a serial number if this is necessary to make it unique.

References

- [CCN⁺85] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. *Atlas of finite groups*. Oxford University Press, Eynsham, 1985. Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray. [9](#)
- [HP89] D. F. Holt and W. Plesken. *Perfect groups*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1989. With an appendix by W. Hanrath, Oxford Science Publications. [4](#), [5](#), [6](#), [7](#), [9](#)
- [Hul22] A. Hulpke. The perfect groups of order up to two million. *Math. Comp.*, 91(334):1007–1017, 2022. [4](#)

Index

DisplayInformationPerfectGroups

for a pair [order, index], [7](#)

for group order (and index), [7](#)

extraperfect, [8](#)

NrPerfectGroups, [5](#)

NrPerfectLibraryGroups, [5](#)

NumberPerfectGroups, [5](#)

NumberPerfectLibraryGroups, [5](#)

perfect groups, [4](#)

PerfectGroup

for a pair [order, index], [5](#)

for group order (and index), [5](#)

PerfectIdentification, [5](#)

SizeNumbersPerfectGroups, [6](#)

SizesPerfectGroups, [4](#)