The Atomic Theory as Applied To Gases, with Some Experiments on the Viscosity of Air

by

Silas W. Holman

Submitted to the Department of Physics
in partial fulfillment of the requirements for the degree of

BACHELOR OF SCIENCE IN PHYSICS

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1876

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ABSTRACT

The developments of the "kinetic theory" of gases made within the last ten years have enabled it to account satisfactorily for many of the laws of gases. The mathematical deductions of Clausius, Maxwell and others, based upon the hypothesis of a gas composed of molecules acting upon each other at impact like perfectly elastic spheres, have furnished expressions for the laws of its elasticity, viscosity, conductivity for heat, diffusive power and other properties. For some of these laws we have experimental data of value in testing the validity of these deductions and assumptions. Next to the elasticity, perhaps the phenomena of the viscosity of gases are best adapted to investigation.[1]

Thesis supervisor: Edward C. Pickering
Title: Professor of Physics

[1] Text from Holman (1876): doi:10.2307/25138434.

Acknowledgments

Write your acknowledgments here.

Biographical Sketch

Silas Whitcomb Holman was born in Harvard, Massachusetts on January 20, 1856. He received his S.B. degree in Physics from MIT in 1876, and then joined the MIT Department of Physics as an Assistant. He became Instructor in Physics in 1880, Assistant Professor in 1882, Associate Professor in 1885, and Full Professor in 1893. Throughout this period, he struggled with increasingly severe rheumatoid arthritis. At length, he was defeated, becoming Professor Emeritus in 1897 and dying on April 1, 1900.

Holman's light burned brilliantly before his tragic and untimely death. He published extensively in thermal physics, and authored textbooks on precision measurement, fundamental mechanics, and other subjects. He established the original Heat Measurements Laboratory. Holman was a much admired teacher among both his students and his colleagues. The reports of his department and of the Institute itself refer to him frequently in the 1880's and 1890's, in tones that gradually shift from the greatest respect to the deepest sympathy.

Holman was a student of Professor Edward C. Pickering, then head of the Physics department. Holman himself became second in command of Physics, under Professor Charles R. Cross, some years later. Among Holman's students, several went on to distinguish themselves, including: the astronomer George E. Hale ('90) who organized the Yerkes and Mt. Wilson observatories and who designed the 200 inch telescope on Mt. Palomar; Charles G. Abbot ('94), also an astrophysicist and later Secretary of the Smithsonian Institution; and George K. Burgess ('96), later Director of the Bureau of Standards.

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Chapter 1

Introduction

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1.1 A section discussing the first issue: J/ψ

We begin with some ideas from the literature [6,7].

$$\frac{\partial}{\partial t}\left[\rho\left(e+|\vec{u}|^2/2\right)\right]+\nabla\cdot\left[\rho\left(h+|\vec{u}|^2/2\right)\vec{u}\right]=-\nabla\cdot\vec{q}+\rho\vec{u}\cdot\vec{g}+\frac{\partial}{\partial x_j}\left(d_{ji}u_i\right) \tag{1.1}$$

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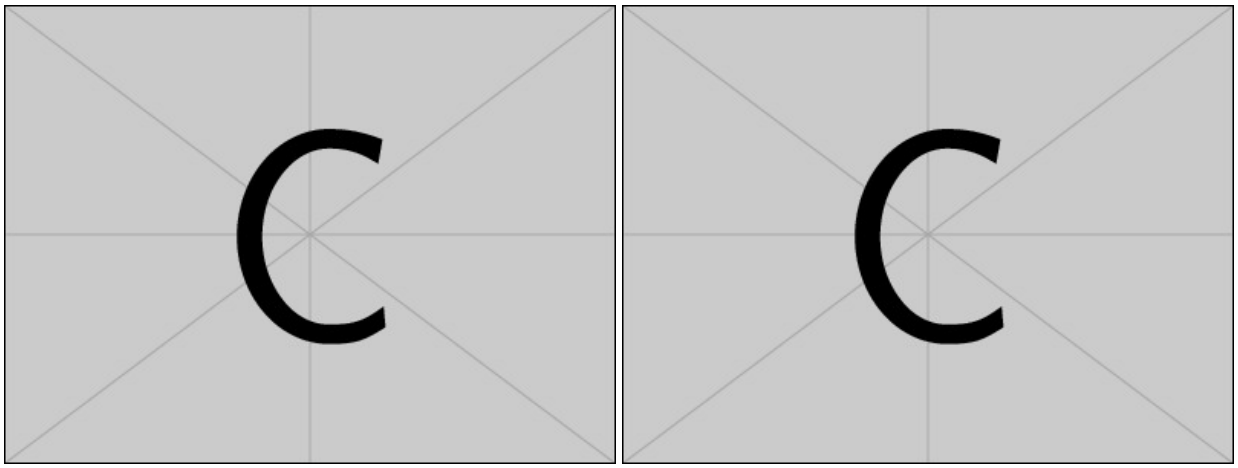

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(b) Second subfigure

Figure 1.1: A figure with two subfigures

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1.1.1 Subsection eqn. (1.2)

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$$L(\mathbf{A}) = \begin{pmatrix} \frac{\varphi}{(\varphi_1, \varepsilon_1)} & 0 & \cdots & \cdots & \cdots & 0 \\ \frac{\varphi k_{2,1}}{(\varphi_2, \varepsilon_1)} & \frac{\varphi}{(\varphi_2, \varepsilon_2)} & 0 & \cdots & \cdots & 0 \\ \frac{\varphi k_{3,1}}{(\varphi_3, \varepsilon_1)} & \frac{\varphi k_{3,2}}{(\varphi_3, \varepsilon_2)} & \frac{\varphi}{(\varphi_3, \varepsilon_3)} & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ \frac{\varphi k_{n-1,1}}{(\varphi_{n-1}, \varepsilon_1)} & \frac{\varphi k_{n-1,2}}{(\varphi_{n-1}, \varepsilon_2)} & \cdots & \frac{\varphi k_{n-1,n-2}}{(\varphi_{n-1}, \varepsilon_{n-2})} & \frac{\varphi}{(\varphi_{n-1}, \varepsilon_{n-1})} & 0 \\ \frac{\varphi k_{n,1}}{(\varphi_n, \varepsilon_1)} & \frac{\varphi k_{n,2}}{(\varphi_n, \varepsilon_2)} & \cdots & \cdots & \frac{\varphi k_{n,n-1}}{(\varphi_n, \varepsilon_{n-1})} & \frac{\varphi}{(\varphi_n, \varepsilon_n)} \end{pmatrix} \tag{1.2}$$

1.2 Description of our paradigm

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[1] First footnote: $a_h = F_m$. See section 1.4.

[2] Another interesting detail.

[3] And another really important idea to have in mind [12–17].

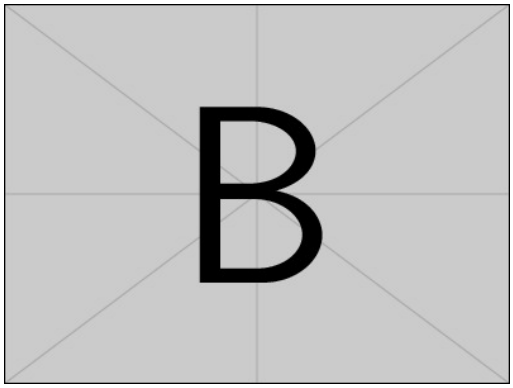

Figure 1.2: Caption text [8].

1.2.1 Conversion to a metaheuristic

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Subsection

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1.3 Other generalizations

Sed commodo posuere pede. Mauris ut est. Ut quis purus. Sed ac odio. Sed vehicula hendrerit sem. Duis non odio. Morbi ut dui. Sed accumsan risus eget odio. In hac habitasse platea dictumst. Pellentesque non elit. Fusce sed justo eu urna porta tincidunt. Mauris felis odio, sollicitudin sed, volutpat a, ornare ac, erat. Morbi quis dolor. Donec pellentesque, erat ac sagittis semper, nunc dui lobortis purus, quis congue purus metus ultricies tellus. Proin et quam. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Praesent sapien turpis, fermentum vel, eleifend faucibus, vehicula eu, lacus. And another citation, so that our sources will be unambiguous [20].

1.3.1 The most general case

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vestibulum ut, massa. Fusce ultrices nulla et nisl. But, according to Lord Brouncker (1655),

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cfrac{9^2}{2 + \cfrac{}{2 + \cdots}}}}}} \tag{1.3}$$

1.4 Baroclinic generation of vorticity

Substitution of the particle acceleration and application of Stokes theorem leads to the *Kelvin–Bjerknes circulation theorem*, for $\rho \neq \mathrm{fn}(p)$:

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\oint_C \mathbf{u}\cdot d\mathbf{r} \tag{1.4}$$

$$= \oint_C \frac{D\mathbf{u}}{Dt}\cdot d\mathbf{r} + \oint_C \mathbf{u}\cdot \underbrace{d\left(\frac{d\mathbf{r}}{dt}\right)}_{=0} \tag{1.5}$$

$$= \iint_S \nabla\times\frac{D\mathbf{u}}{Dt}\cdot d\mathbf{A} \tag{1.6}$$

$$= \iint_S \nabla p\times\nabla\left(\frac{1}{\rho}\right)\cdot d\mathbf{A} \tag{1.7}$$

Baroclinic generation of vorticity accounts for the sea breeze and various other atmospheric currents in which temperature, rather than pressure, creates density gradients. Further, this phenomenon accounts for ocean currents in straits joining more and less saline seas, with surface currents flowing from the fresher to the saltier water and with bottom currents going oppositely.

Weak shock wave. A weak shock wave travelling in a perfect gas is described by the following Burgers equation derived from approximations due to G. I. Taylor (1910)

$$u_t + \left(a_0 + \tfrac{1}{2}(\gamma+1)u\right)u_x = \frac{\delta}{2}u_{xx} \tag{1.8}$$

Here, γ is the specific heat capacity ratio, δ is the diffusivity of sound, u_1 is the downstream speed, $u_2 < u_1$ is the upstream speed, and a_0 is the undisturbed sound speed. The travelling-wave, or Taylor shock, solution of this equation is

$$\frac{u-u_2}{u_1-u_2} = \frac{1}{2}\left[1 - \tanh\left(\frac{\gamma+1}{4\delta}(u_1-u_2)(x-ct)\right)\right] \tag{1.9}$$

where $c = a_0 + (\gamma+1)(u_1+u_2)/4$. This solution exists for a compressive wave, $u_1 > u_2$.

Table 1.1: The error function and complementary error function

x	$\operatorname{erf}(x)$	$\operatorname{erfc}(x)$	x	$\operatorname{erf}(x)$	$\operatorname{erfc}(x)$
0.00	0.00000	1.00000	1.10	0.88021	0.11979
0.05	0.05637	0.94363	1.20	0.91031	0.08969
0.10	0.11246	0.88754	1.30	0.93401	0.06599
0.15	0.16800	0.83200	1.40	0.95229	0.04771
0.20	0.22270	0.77730	1.50	0.96611	0.03389
0.30	0.32863	0.67137	1.60	0.97635	0.02365
0.40	0.42839	0.57161	1.70	0.98379	0.01621
0.50	0.52050	0.47950	1.80	0.98909	0.01091
0.60	0.60386	0.39614	1.8214	0.99000	0.01000
0.70	0.67780	0.32220	1.90	0.99279	0.00721
0.80	0.74210	0.25790	2.00	0.99532	0.00468
0.90	0.79691	0.20309	2.50	0.99959	0.00041
1.00	0.84270	0.15730	3.00	0.99998	0.00002

Table 1.2: Table with more complicated columns: $\Delta T_m = \delta\theta/\Theta_0$

Experiment	u [m/s]	T [°C]
The first test we ran this morning	124.3	68.3
The second test we ran this morning	82.50	103.46
Our competitor's test	72.321	141.384

Boundary layer model. K. O. Friedrichs introduced the following model equation to illustrate a boundary layer (1941):

$$\varepsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} = 1 + 2x, \quad \text{with } y(0) = 0 \text{ and } y(1) = 1 \tag{1.10}$$

As $\varepsilon \to 0$, the highest derivative is lost: the equation becomes first order and cannot meet two boundary conditions. This fact results in a boundary layer at one side of the domain, with a thickness $\mathcal{O}(\varepsilon)$. Matching the outer and inner expansions gives the uniform approximation

$$y_{\text{unif}} = e^{-x/\varepsilon} - 1 + x + x^2 + 2\varepsilon\big[(1-x) - e^{-x/\varepsilon}\big], \tag{1.11}$$

with $y - y_{\text{unif}} = \mathcal{O}(\varepsilon^3)$ uniformly on $0 \le x \le 1$ as $\varepsilon \to 0$. As it happens, all higher-order terms of both expansions vanish, so y_{unif} is correct to all orders in ε. A comparison to the exact solution shows the error to be subdominant, $\mathcal{O}(e^{-1/\varepsilon})$.

Table 1.3: Brinkman number at which liquid dissipation makes $< 5\%$ or $< 1\%$ contribution to θ_b or q_w from the Graetz solution at the points where θ_b equals 0.05, 0.02, or 0.01

			|Br|			
θ_b	x^+	$q_w D_h/k\Delta T$	5% of θ_b	1% of θ_b	5% of q_w	1% of q_w
0.05	0.09621	0.3770	3.646×10^{-3}	7.292×10^{-4}	1.571×10^{-3}	3.142×10^{-4}
0.02	0.1266	0.1508	1.458×10^{-3}	2.917×10^{-4}	6.284×10^{-4}	1.257×10^{-4}
0.01	0.1496	0.07541	7.292×10^{-4}	1.458×10^{-4}	3.142×10^{-4}	6.284×10^{-5}

1.5 Further examples of multipanel figures

Examples are given in Figs. 1.3, 1.4, and 1.5. Or consider Figs. 1.4(a), 1.4(b), 1.4(c), and 1.4(d).

1.6 Summary

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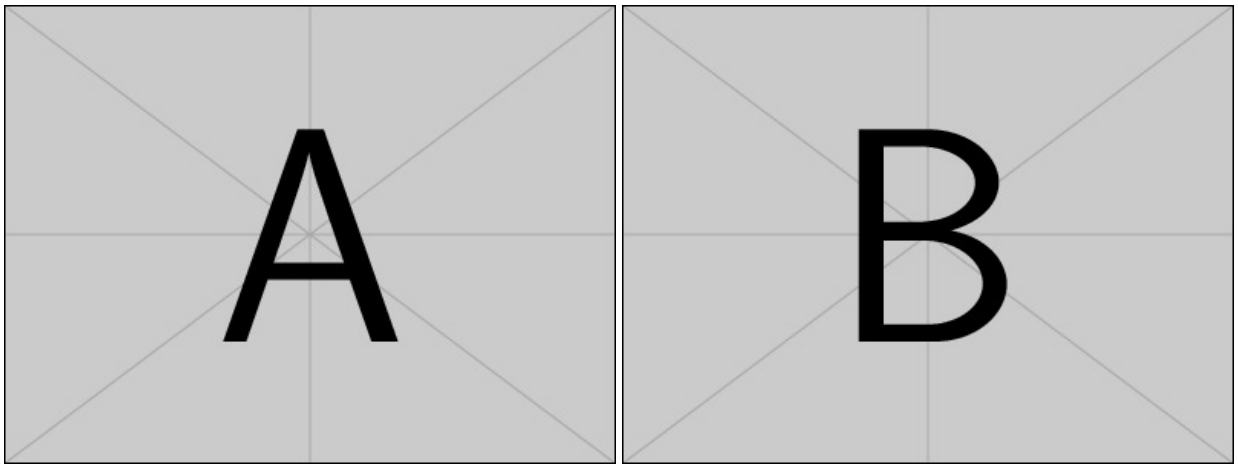
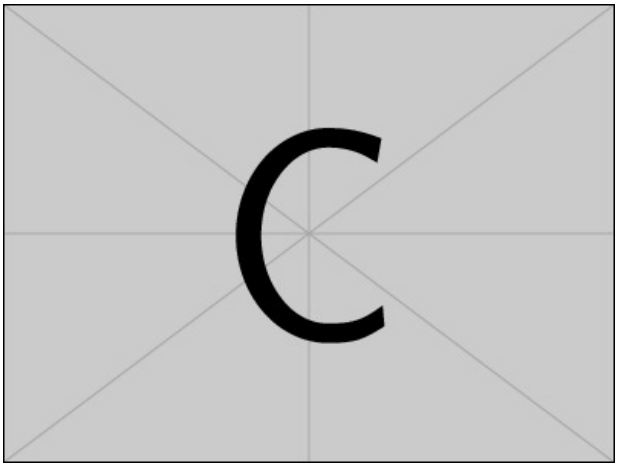


(a) Postremo aliquos futuros suspicor, qui me ad alias litteras vocent, genus hoc scribendi.

(b) Short caption



(c) Caption with math, $\mathrm{Re}_d = 100$, and a reference to Fig. 1.3(a).

Figure 1.3: A figure with three subfigures: two above one.

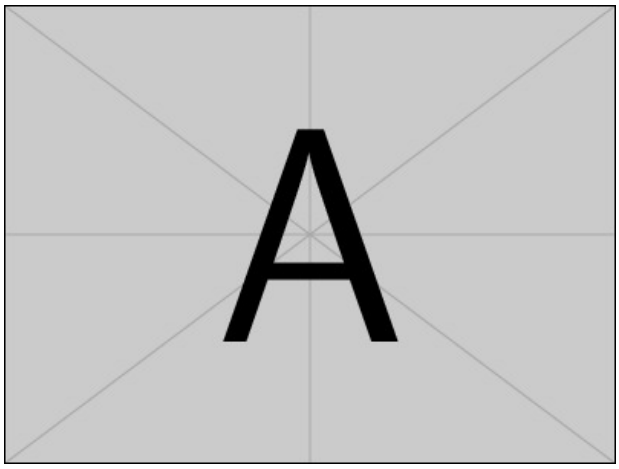
Nomenclature for Chapter 1

Roman letters

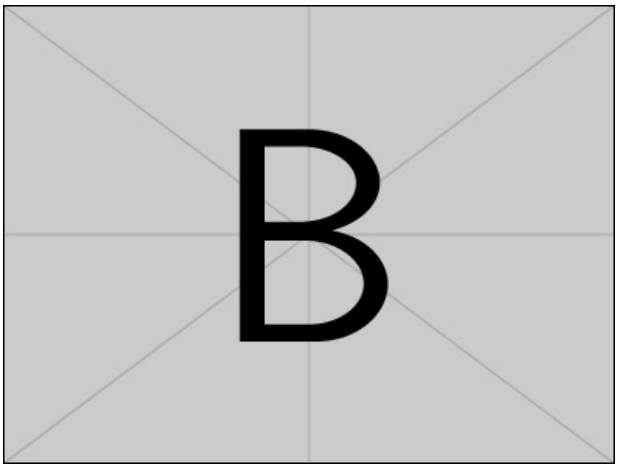
$\mathcal{C}$	material curve
$\mathbf{r}$	material position [m]
$\mathbf{u}$	velocity [m s^{-1}]

Greek letters

Γ	circulation [m^2 s^{-1}]
ρ	mass density [kg m^{-3}]
$\boldsymbol{\omega}$	vorticity [s^{-1}]



(a) First subfigure



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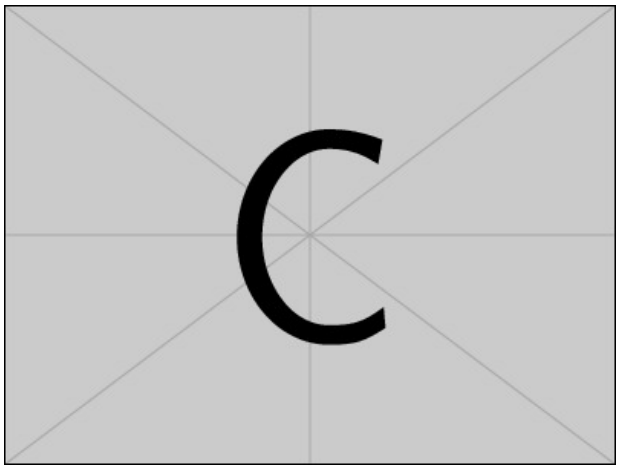

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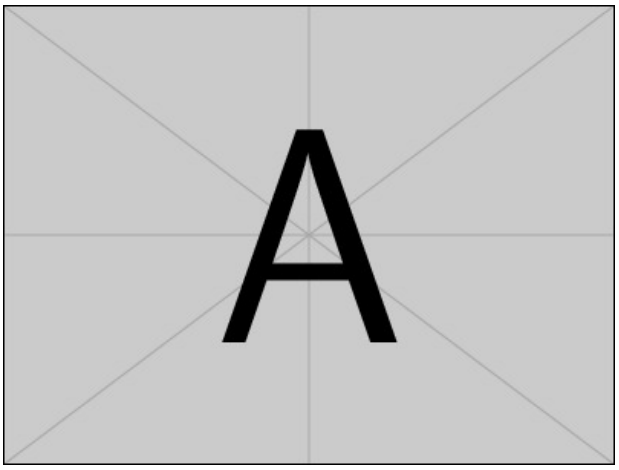
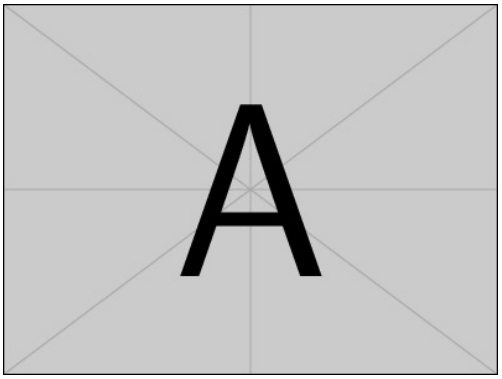
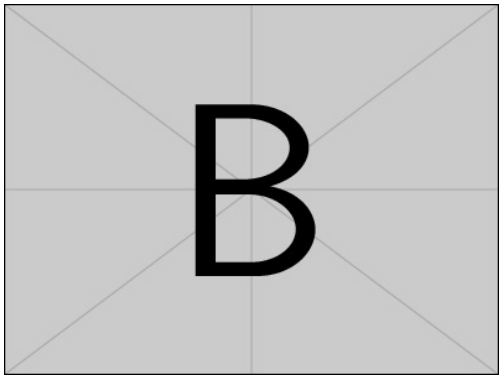


(d) Fourth subfigure, with $\alpha = 0.5$.

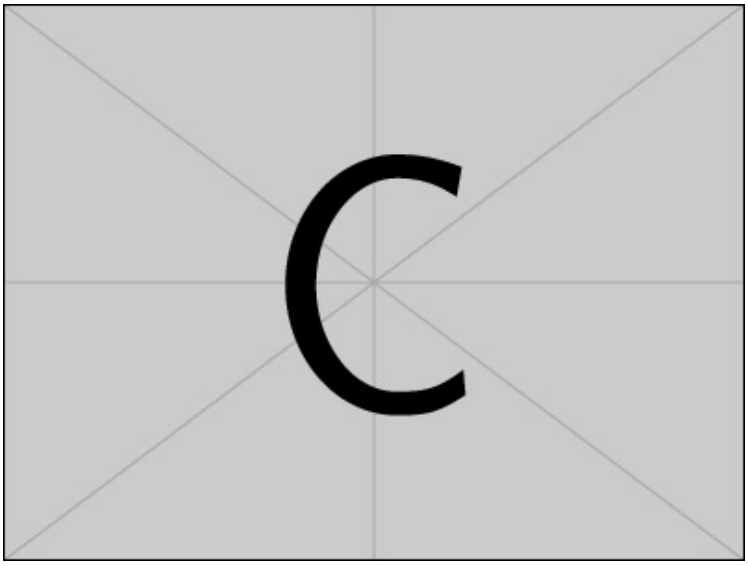
Figure 1.4: A figure with four subfigures: two above two.



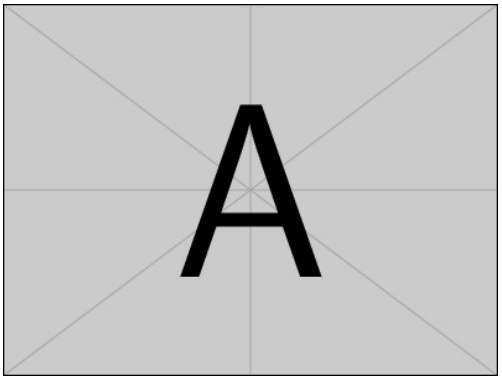
(a) Top left



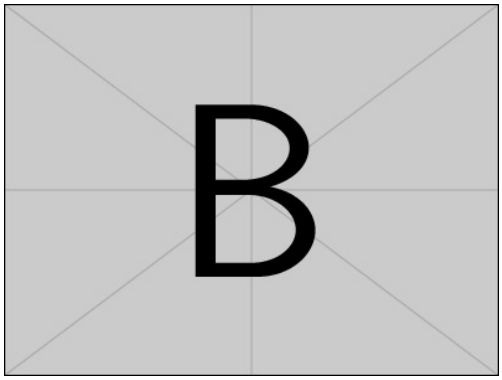
(b) Top right, with a longer caption: postremo aliquos futuros suspicor, qui me ad alias litteras vocent.



(c) The larger middle subfigure, which refers to Figs. 1.5(a) and 1.5(e).



(d) Bottom left, $x^2 + y^2 = r^2$.



(e) Bottom right

Figure 1.5: A figure with five subfigures: two above, one larger in the middle, and two below.

Chapter 2

Various Test Examples

2.1 Some tables: $u_r^2 \cdot \omega_0^\epsilon = 0$

Consider Tables 2.1, 2.2, 2.3, and decide what you think. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

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Table 2.1: A simple table with math in the caption: $z = r\cos\theta, \zeta^2 = e^{i\pi\gamma_n^3}$

Experiment	u [m/s]	T [°C]
Run 11	12.5	103.4
Run 12	24	68.3

Table 2.2: Table with more complicated columns and more math: $\Delta T_m = \delta\theta/\Theta_0$

Experiment	u [m/s]	T [°C]
The first test we ran this morning	124.3	68.3
The second test we ran this morning	82.50	103.46
Our competitor's test	72.321	141.384

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2.2 Some lists $\mathbf{F} = \boldsymbol{A} \otimes \boldsymbol{\Omega}$

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Table 2.3: Brinkman number at which liquid dissipation makes less than a 5% or 1% contribution to θ_b or q_w from the Graetz solution at the points where θ_b equals 0.05, 0.02, or 0.01.

			|Br|			
θ_b	x^+	$q_w D_h/k\Delta T$	5% of θ_b	1% of θ_b	5% of q_w	1% of q_w
0.05	0.09621	0.3770	3.646×10^{-3}	7.292×10^{-4}	1.571×10^{-3}	3.142×10^{-4}
0.02	0.1266	0.1508	1.458×10^{-3}	2.917×10^{-4}	6.284×10^{-4}	1.257×10^{-4}
0.01	0.1496	0.07541	7.292×10^{-4}	1.458×10^{-4}	3.142×10^{-4}	6.284×10^{-5}

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$$\hat{x}_i \hat{x}_j = \hat{x}_j \hat{x}_i, \quad \hat{x}_i^2 = 0, \quad i, j = 1, \dots, n.$$

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Table 2.4: External packages loaded by `mitthesis.cls`. For documentation, visit CTAN, https://ctan.org. Or, if you have TeX Live installed, you can open a terminal window and type `% texdoc package-name`.

Package	Purpose in class file	How it can be used
`bookmark`	manages PDF bookmarks (is loaded by `hyperref` if `\DocumentMetadata` is active)	customize PDF bookmarks
`doi`	support for hyperlinking doi number	hyperlink a doi number: `\doi{..}`
`etoolbox`	extend or modify other macros	can use in preamble to modify macros, if needed
`geometry`	set page size and margins	can use `\newgeometry` in `mydesign.tex` to change margins
`graphicx`	support for inserting images	use to include graphics
`hyperref`	support for hyperlinks and metadata	add metadata for thesis in preamble
`mathtools`	loads and extends `amsmath`	**adds many useful math macros.** See documentation for `amsmath` and `mathtools`
`lineno`	class option to add line numbers	class option `[lineno]` will give line numbers; `lineno` package includes commands to control numbering
`bm`	enable bold math symbols with pdfTeX	command `\bm{..}` produces a bold math symbol
`fontenc`	manages font encoding with pdfTeX	additional encodings can be set for non-Latin fonts in a fontset file
`fontspec`	required for text font management with LuaLaTeX (`unicode-math` loads `fontspec` by default)	use to load additional Unicode text fonts
`unicode-math`	required for math font management with LuaLaTeX	can choose math symbols, e.g. `\symbfit{\Gamma}` for $\boldsymbol{\Gamma}$; load or alter math fonts in a fontset file
`xcolor`	support for colors, including colored fonts	`\textcolor{color}{text}`, `\color{color name}`

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Boundary layer model. K. O. Friedrichs introduced the following model equation to illustrate a boundary layer (1941):

$$\varepsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} = 1 + 2x, \quad \text{with } y(0) = 0 \text{ and } y(1) = 1 \tag{2.1}$$

As $\varepsilon \to 0$, the highest derivative is lost: the equation becomes first order and cannot meet two boundary conditions. This fact results in a boundary layer at one side of the domain, with a thickness $\mathcal{O}(\varepsilon)$. Matching the outer and inner expansions gives the uniform approximation

$$y_{\text{unif}} = e^{-x/\varepsilon} - 1 + x + x^2 + 2\varepsilon\big[(1 - x) - e^{-x/\varepsilon}\big], \tag{2.2}$$

with $y - y_{\text{unif}} = \mathcal{O}(\varepsilon^3)$ uniformly on $0 \le x \le 1$ as $\varepsilon \to 0$. As it happens, all higher-order terms of both expansions vanish, so y_{unif} is correct to all orders in ε. A comparison to the exact solution shows the error to be subdominant, $\mathcal{O}(e^{-1/\varepsilon})$.

2.3 Enumeration of Hamiltonian paths in a graph

Let $\mathbf{A} = (a_{ij})$ be the adjacency matrix of graph G. The corresponding Kirchhoff matrix $\mathbf{K} = (k_{ij})$ is obtained from $\mathbf{A}$ by replacing in $-\mathbf{A}$ each diagonal entry by the degree of its corresponding vertex; i.e., the ith diagonal entry is identified with the degree of the ith vertex. It is well known that

$$\det \mathbf{K}(i|i) = \text{ the number of spanning trees of } G, \quad i = 1, \dots, n \tag{2.3}$$

where $\mathbf{K}(i|i)$ is the ith principal submatrix of $\mathbf{K}$.

```
\det\mathbf{K}(i|i)=\textrm{ the number of spanning trees of $G$},
```

Let $C_{i(j)}$ be the set of graphs obtained from G by attaching edge $(v_i v_j)$ to each spanning tree of G. Denote by $C_i = \bigcup_j C_{i(j)}$. It is obvious that the collection of Hamiltonian cycles is a subset of C_i. Note that the cardinality of C_i is $k_{ii} \det \mathbf{K}(i|i)$. Let $X = \{\hat{x}_1, \dots, \hat{x}_n\}$.

```
$ X=\{\hat x_1,\dots,\hat x_n\}$
```

Define multiplication for the elements of X by

$$\hat{x}_i \hat{x}_j = \hat{x}_j \hat{x}_i, \quad \hat{x}_i^2 = 0, \quad i, j = 1, \dots, n. \tag{2.4}$$

Let $\hat{k}_{ij} = k_{ij}\hat{x}_j$ and $\hat{k}_{ij} = -\sum_{j \neq i} \hat{k}_{ij}$. Then the number of Hamiltonian cycles H_c is given by the relation

$$\left(\prod_{j=1}^{n} \hat{x}_j\right) H_c = \frac{1}{2} \hat{k}_{ij} \det \mathbf{K}(i|i), \qquad i = 1, \dots, n. \tag{2.5}$$

The task here is to express (2.5) in a form free of any $\hat{x}_i$, $i = 1, \dots, n$. The result also leads to the resolution of enumeration of Hamiltonian paths in a graph.

A function $H : \mathfrak{R}^n \to \mathfrak{R}^n$ is said to be *B-differentiable* at the point z if (i) H is Lipschitz continuous in a neighborhood of z, and (ii) there exists a positive homogeneous function $BH(z) : \mathfrak{R}^n \to \mathfrak{R}^n$, called the *B-derivative* of H at z, such that

$$\lim_{v \to 0} \frac{H(z+v) - H(z) - BH(z)v}{|v|} = 0.$$

The function H is *B-differentiable in set* S if it is B-differentiable at every point in S. The B-derivative $BH(z)$ is said to be *strong* if

$$\lim_{(v,v') \to (0,0)} \frac{H(z+v) - H(z+v') - BH(z)(v-v')}{|v - v'|} = 0.$$

2.3.1 Bold math?

The next two equations will be bold if a bold version of the math font exists.

$$(\prod_{j=1}^{n} \hat{x}_j) H_c = \frac{1}{2} \hat{k}_{ij} \det \mathbf{K}(i|i), \qquad i = 1, \dots, n. \tag{2.6}$$

$$\tan x = \frac{\sin x}{\cos x} \tag{2.7}$$

Some lazy math superscripts. We have m^{-2}, K^{-1}, s^{-1} and some other things.
Similar to a nomenclature issue: [m^2] [m^2] [ω]

Some integrals. Consider this integral:

$$\int_0^\infty \exp(-\varepsilon x^2)\,dx \Big|_{\varepsilon\to 0}$$

Also consider these (the first integral has its limit moved by `\mkern-20mu`):

$$\iint_A f(x,y)\,dx\,dy \qquad \iiint_A f(x,y,z)\,dx\,dy\,dz \tag{2.8}$$

$$\iiiint_A f(w,x,y,z)\,dw\,dx\,dy\,dz \qquad \int_A \cdots \int f(x_1,\ldots,x_k) \tag{2.9}$$

$$\overrightarrow{\psi_\delta(t)E_t h} = \underrightarrow{\psi_\delta(t)E_t h}$$
$$\overleftarrow{\psi_\delta(t)E_t h} = \underleftarrow{\psi_\delta(t)E_t h}$$
$$\overleftrightarrow{\psi_\delta(t)E_t h} = \underleftrightarrow{\psi_\delta(t)E_t h}$$

2.3.2 `overset`, `underset`, and `sideset`

Examples:

$$\overset{*}{X} \qquad \underset{*}{X} \qquad \overset{a}{\underset{b}{X}}$$

```
\[\overset{*}{X}\qquad\underset{*}{X}\qquad
\overset{a}{\underset{b}{X}}\]
%
```

The command `sideset` is for a rather special purpose: putting symbols at the subscript and superscript corners of a large operator symbol such as $\sum$ or $\prod$, without affecting the placement of limits. Examples:

$$\overset{*}{\underset{*}{\prod_k}}\overset{*}{\underset{*}{}} \qquad \sum_{0\le i\le m}' E_i\beta x$$

```
\[\sideset{_*^*}{_*^*}\prod_k\qquad
\sideset{}{'}\sum_{0\le i\le m} E_i\beta x
\]
```

2.3.3 Continued fractions

The continued fraction

$$\cfrac{1}{\sqrt{2}+\cfrac{1}{\sqrt{2}+\cfrac{1}{\sqrt{2}+\cfrac{1}{\sqrt{2}+\cfrac{1}{\sqrt{2}+\cdots}}}}} \tag{2.10}$$

can be obtained by typing

```
\cfrac{1}{\sqrt{2}+
 \cfrac{1}{\sqrt{2}+
  \cfrac{1}{\sqrt{2}+
   \cfrac{1}{\sqrt{2}+
    \cfrac{1}{\sqrt{2}+\dotsb
}}}}}
```

$$\begin{matrix} \vartheta & \varrho \\ \varphi & \varpi \end{matrix} \quad \begin{pmatrix} \vartheta & \varrho \\ \varphi & \varpi \end{pmatrix} \quad \begin{bmatrix} \vartheta & \varrho \\ \varphi & \varpi \end{bmatrix} \quad \begin{Bmatrix} \vartheta & \varrho \\ \varphi & \varpi \end{Bmatrix} \quad \begin{vmatrix} \vartheta & \varrho \\ \varphi & \varpi \end{vmatrix} \quad \begin{Vmatrix} \vartheta & \varrho \\ \varphi & \varpi \end{Vmatrix} \tag{2.11}$$

2.4 Code lines set using `verbatim`

```
\cs_new_protected:Nn \__mitthesis_gset_author_data:nnnnnn
 {
 \int_gincr:N \g__mitthesis_author_cnt_int
 \seq_gput_right:Ne  \g__mitthesis_author_name_seq             { \tl_trim_spaces:n {#1} }
 \seq_gput_right:Ne  \g__mitthesis_author_department_seq       { \tl_trim_spaces:n {#2} }
 \seq_gput_right:Ne  \g__mitthesis_author_prevdegree_seq       { \tl_trim_spaces:n {#3} }
 \seq_gput_right:Ne  \g__mitthesis_author_secondprevdegree_seq { \tl_trim_spaces:n {#4} }
 \seq_gput_right:Ne  \g__mitthesis_author_thirdprevdegree_seq  { \tl_trim_spaces:n {#5} }
 \seq_gput_right:Ne  \g__mitthesis_author_fourthprevdegree_seq { \tl_trim_spaces:n {#6} }
 }

\NewExpandableDocumentCommand\CopyrightAuthor{}{
 \tl_if_exist:NTF \g__mitthesis_copyright_author_tl
  { \g__mitthesis_copyright_author_tl }
  { \__mitthesis_gset_copyright_author: \g__mitthesis_copyright_author_tl }
}
```

2.5 Introduction

A recent paper has claimed that the Nusselt number for thermally developed flow between plates is not 7.541, as most textbooks report . Instead, the paper recommends the values

35/2 (incompressible liquids) and 0 (ideal gases). The rationale is that flow work and/or dissipation are inevitably present in viscous flow, causing heating or cooling in the bulk which alters the final heat flux far downstream. This suggestion is in fact not new, having also been made for gas flows in 1973 and discussed (but not adopted) by others in 1978 . While the physics and calculations of these papers are not wrong, the claim fails to consider the relative size of such effects in the region where the heat load is removed after a wall temperature change. The latter is what normally interests engineers.

Here, I show that the classical Graetz solution, $\mathrm{Nu}_{D_h} = 7.541$, is appropriate for calculating heat transfer in gases and liquids. The heat transfer coefficient should not be defined to include dissipative effects, which are independent of the local wall to bulk temperature difference. Superposition shows that dissipative effects can simply be added to the Graetz values, if those effects are not entirely negligible. Further, I note that the Seban-Shimazaki criterion is misleading when dissipation is considered. In the presence of flow work and/or dissipation, thermally developed flow is simply the one-term limit of the Graetz series, just as in the nondissipative analysis.

2.6 Governing equations

Two parallel plates are a distance H apart, with the y coordinate centered in the channel. The bulk velocity is u_b and the velocity profile for fully developed hydrodynamics is

$$u(y) = \frac{3}{2} u_b \left[1 - 4\left(\frac{y}{H}\right)^2\right] \tag{2.12}$$

with pressure gradient

$$-\frac{dp}{dx} = \frac{12\mu u_b}{H^2} \tag{2.13}$$

The steady thermal energy equation[1] for a Newtonian fluid is

$$\rho c_p \mathbf{u} \cdot \nabla T = -\nabla \cdot \mathbf{q} + \beta T\, \mathbf{u} \cdot \nabla p + d_{ji} \frac{\partial u_i}{\partial x_j} \tag{2.14}$$

where symbols are standard (see nomenclature). For fully developed hydrodynamics and neglecting axial heat conduction, this equation reduces to

$$\rho c_p u(y) \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial y^2} + \beta T\, u(y) \frac{dp}{dx} + \mu \left(\frac{du}{dy}\right)^2 \tag{2.15}$$

For $x < 0$, the wall is at T_0 and for $x \geq 0$, $T(x, \pm H/2) = T_w$. Dissipation is present both before and after the wall temperature changes. Accordingly, the initial condition at $x = 0$ must be the solution for a thermally fully developed flow with the wall at T_0 in the presence of flow work and/or dissipation.

[1] The thermal energy equation ($dh = T\, ds + v\, dp$) is obtained from the [total] energy equation by deducting the dot product of the momentum equation and the velocity. This equation does not neglect kinetic energy change, but axial density change is not included in the classical Graetz solution (pressure gradients are treated as thermodynamically small). That has the effect of neglecting kinetic energy changes.

The classical Graetz problem is described by a balance between the first two terms in Eq. (2.15). Far downstream of the wall temperature change, the first term diminishes to zero, while flow work and dissipation continue to provide an unvarying source of heating or cooling within the channel. Thus, upstream of the wall temperature change, and far downstream of it, the three right-hand side terms describe heat transfer to the channel wall. Here, I ask when those effects may be ignored in comparison to the heat transfer caused by the wall temperature change and how they should be represented if they are non-negligible.

2.7 Solution by superposition

Equation (2.15) is linear in $T(x,y)$ and therefore amenable to superposition. Let $T(x,y) = T_d + T_{nd}$, where T_d takes the source (dissipation) term with an isothermal wall at $T_d = 0$ and T_{nd} has no dissipative term. The dissipative solution, T_d, describes the temperature distribution for fully developed flow with $T(x, \pm H/2) = 0$. The nondissipative solution is the classical Graetz solution without flow work and/or dissipation.

Variants of this analysis have already appeared in the literature, even 50 years ago. The purpose of this section is simply to set down, in consistent notation, the equations needed for the main question: does dissipation invalidate the classical Nusselt number of the thermally developed Graetz solution?

2.7.1 Dissipative solution

Conditions do not vary in x, so the first term in Eq. (2.15) is zero. The temperature field is $T_d(y)$. The nondimensionalization is as follows: $y = H\bar{y}$, $u = u_b\bar{u}$, and $\theta_d = T_d/\Delta T$, where for the present ΔT is arbitrary. Then

$$\frac{d^2\theta_d}{d\bar{y}^2} = \mathrm{Br}\left[12(\beta T)\bar{u}(\bar{y}) - \left(\frac{d\bar{u}}{d\bar{y}}\right)^2\right] \tag{2.16a}$$

$$= \mathrm{Br}\left[18(\beta T)(1-4\bar{y}^2) - 144\bar{y}^2\right] \tag{2.16b}$$

subject to $\theta_d(\bar{y} = \pm 1/2) = 0$. Here $\mathrm{Br} = \mu u_b^2/k\Delta T$. The Brinkman number may be positive or negative, according to the sign of the temperature difference.

The product βT will be nearly constant over the cross-section for typical values of Br. Integrating, with $d\theta/d\bar{y} = 0$ at $\bar{y} = 0$,

$$\frac{d\theta_d}{d\bar{y}} = \mathrm{Br}\left[18(\beta T)\left(\bar{y} - \frac{4}{3}\bar{y}^3\right) - 48\bar{y}^3\right] \tag{2.17a}$$

$$\theta_d = \mathrm{Br}\underbrace{\left[9(\beta T)\left(\bar{y}^2 - \frac{2}{3}\bar{y}^4\right) - 12\bar{y}^4 - \frac{15}{8}(\beta T) + \frac{3}{4}\right]}_{=f(\bar{y})} \tag{2.17b}$$

The bulk temperature is

$$\theta_{d,b} = \int_{-1/2}^{+1/2} \bar{u}(\bar{y})\theta_d(\bar{y})\, d\bar{y} = \mathrm{Br}\left[\underbrace{-\frac{51}{35}(\beta T)}_{\equiv a} + \frac{24}{35}\right] \tag{2.18a}$$

$$= \begin{cases} +\dfrac{24}{35}\,\mathrm{Br} & \beta T = 0, \text{ incompressible liquid} \\ -\dfrac{27}{35}\,\mathrm{Br} & \beta T = 1, \text{ ideal gas} \end{cases} \tag{2.18b}$$

When $\mathrm{Br} \to 0$, $\theta_{d,b} \to 0$, or $T_d = T_{d,b} = 0$.

These results are well-known in the literature . Consider $\mathrm{Br} > 0$. In the first instance, dissipation makes the bulk temperature greater than the wall temperature. In the second instance, flow work lowers the bulk temperature more strongly than dissipation raises it.

The dimensionless temperature gradient at the wall is

$$\left.\frac{d\theta_d}{d\bar{y}}\right|_{\bar{y}=+H/2} = 6\mathrm{Br}\,(\beta T - 1) \tag{2.19}$$

For an ideal gas ($\beta T = 1$), the combination of flow work and dissipation produces *zero wall heat flux*.

2.7.2 Graetz solution for parallel plates

The Graetz problem between parallel plates has been studied exhaustively . A snapshot of relevant details follows. For a wall temperature changing from T_0 to T_w at $x = 0$, the Graetz solution gives a series

$$\theta_{pp}(x^+, \bar{y}) = \frac{T_{pp} - T_w}{T_0 - T_w}$$
$$= B_0 \exp\left(-\frac{32}{3}\lambda_0^2 x^+\right) Y_0(\bar{y}) + \sum_{n=1}^{\infty} B_n \exp\left(-\frac{32}{3}\lambda_n^2 x^+\right) Y_n(\bar{y}) \text{for } x^+ = \frac{(x/D_h)}{\mathrm{Re}_{D_h}\mathrm{Pr}} \tag{2.20}$$

The first right-hand side term is the thermally developed solution, applicable at large x^+ where the rest of the series has become negligible. The eigenfunctions, $Y_n(\bar{y})$, may be written in terms of the confluent hypergeometric function . The bulk temperature is

$$\theta_{pp,b}(x^+) = 3\sum_{n=0}^{\infty} \frac{G_n}{\lambda_n^2} \exp\left(-\frac{32}{3}\lambda_n^2 x^+\right) \tag{2.21}$$

where the eigenvalues and coefficients for a uniform inlet temperature are given in Table 2.5.

The thermal entry length can be defined as the point at which the local Nu_{D_h} is 5% greater than the thermally developed value: $x_{te}^+ = 0.00797350$. At this point, the second term of the series is just 0.48% of the first term. A single term approximation to Eq. (2.21) suffices in this region:

$$\theta_{pp,b}(x^+) \simeq 3\frac{G_0}{\lambda_0^2} \exp\left(-\frac{32}{3}\lambda_0^2 x^+\right) \quad \text{for } x^+ \geq 0.00797 \tag{2.22}$$

Table 2.5: Eigenvalues and coefficients for Eq. (2.21) . For higher n: $\lambda_n \doteq 4n + 5/3$ and $G_n \doteq 1.01278\,\lambda_n^{-1/3}$.

n	λ_n	G_n
0	1.68159	0.858087
1	5.66985	0.569463
2	9.66842	0.476065

The wall heat flux at $\bar{y} = 1/2$, $q_{w,pp}(x^+)$, is

$$q_{w,pp}(x^+) = \frac{8(T_0 - T_w)k}{D_h} \sum_{n=0}^{\infty} G_n \exp\left(-\frac{32}{3}\lambda_n^2 x^+\right) \tag{2.23}$$

Here, $q_{w,pp} > 0$ for heat leaving the duct, as when $T_w < T_0$. The local heat transfer coefficient, h_{pp}, is found by dividing the wall flux by the bulk-to-wall temperature difference. In the thermally developed region ($x^+ \geq x_{te}^+ = 0.00797$), the Nusselt number is $\mathrm{Nu}_{D_h} = q_w D_h / k\theta_b(T_0 - T_w) = 8\lambda_0^2/3 = 7.5407$, using the first term of each series.

2.7.3 Nondissipative solution

The nondissipative problem is

$$\rho c_p u(y)\frac{\partial T_{nd}}{\partial x} = k\frac{\partial^2 T_{nd}}{\partial y^2} \tag{2.24}$$

subject to $T_{nd}(0,y) = T(0,y) - T_d(y) = T_0$, $T_{nd}(x,\pm H/2) = T_w - T_d(x,\pm H/2) = T_w$. For nondimensionalization, we define

$$\theta_{nd} = \frac{T_{nd} - T_w}{T_0 - T_w} \tag{2.25}$$

with $\theta_{nd}(0,\bar{y}) = 1$ and $\theta_{nd}(x \to \infty, \bar{y}) = 0$. These equations are simply the classical Graetz problem, and so $\theta_{nd} = \theta_{pp}$.

2.7.4 Fully developed solution for θ_b and q_w

Combining the two solutions, and setting $\Delta T = (T_0 - T_w)$, gives

$$\theta = \frac{T - T_w}{T_0 - T_w} = \theta_{nd} + \theta_d = \theta_{pp} + \mathrm{Br}\, f(\bar{y}) \tag{2.26}$$

where $f(\bar{y})$ is defined in Eq. (2.17b). The bulk temperature for $x^+ > x_{te}^+$ is

$$\theta_b = \theta_{pp,b} + a\mathrm{Br} = 3\sum_{n=0}^{\infty} \frac{G_n}{\lambda_n^2} \exp\left(-\frac{32}{3}\lambda_n^2 x^+\right) + a\mathrm{Br} \tag{2.27a}$$

$$\simeq \frac{3G_0}{\lambda_0^2} \exp\left(-\frac{32}{3}\lambda_0^2 x^+\right) + a\mathrm{Br} \tag{2.27b}$$

where a was defined in Eq. (2.18a).

For the case of an ideal gas, $\beta T = 1$, the additional heat flux from flow work and dissipation is exactly zero (Eq. 2.19), leaving only the Graetz solution as a contributor to heat flux. Moreover, if we were to ignore dissipation and flow work entirely in computing the bulk temperature and heat transfer coefficient of an ideal gas flow, we would still compute the heat flux correctly. (The temperature profile would vary by a function $\mathrm{Br}\, f(\bar{y})$, according to eqn. (2.26).[2])

The local heat flux out of the upper plate for an incompressible liquid, with Eqs. (2.19), (2.23), and (2.26), is

$$q_w(x^+) = q_{w,pp}(x^+) + \frac{6k(T_0 - T_w)}{H}\,\mathrm{Br} \tag{2.28}$$

Observe that the last term is positive irrespective of the sign of $T_0 - T_w$, whereas the second is negative when $T_0 - T_w < 0$ (when the wall heats the liquid). Upstream of the wall temperature change, and far downstream of it, the heat generated by viscous dissipation flows out the wall ($q_w > 0$ at the upper plate). When the wall temperature change cools the liquid, the flux out the upper plate is greater, but still positive. When the wall temperature change heats the liquid, heat flows into the liquid ($q_w < 0$) until the heating process is nearly complete, after which heat again flows out. In short, the heat flux changes sign during the liquid heating process.

The dimensionless bulk temperature, θ_b, can also change sign when $\mathrm{Br} < 0$. This behavior simply signifies that $T_b < T_w$ during part of the heating process. However, it has ramifications for the definition of the heat transfer coefficient and for the Nusselt number (Section 2.9).

2.8 When can dissipation be ignored?

The contribution of flow work and/or dissipation to the bulk temperature is the same for every x^+. Specifically, the increase or decrease in bulk temperature is $a\mathrm{Br}$, from Eq. (2.18a). Since a is normally of order one, the effect on bulk temperature will negligible when $|\mathrm{Br}| \ll 1$, until x^+ is large enough that θ_b has decayed to a comparably small size.

For an ideal gas, as previously noted, the heat transfer may be computed accurately by ignoring flow work and dissipation, which cancel from the heat flux. As a result, we consider only liquids in this section.

For $x^+ \to \infty$, the wall heat flux arises only from dissipation and is not zero. However, for a finite duct length, $L_{\max}$, the dissipative heat flux can remain negligible relative to the flux caused by the wall temperature change. We now estimate $L_{\max}$.

We calculate the range $x^+ \le L_{\max}$ within which the thermally developed heat flux with dissipation is within 5% of the heat flux from the full Graetz solution, Eq. (2.23). The wall heat flux for a liquid in the thermally developed region, including dissipation, is

$$q_w(x^+) \simeq \frac{(T_0 - T_w)k}{D_h}\left[8G_0 \exp\left(-32\lambda_0^2 x^+/3\right) + 12\mathrm{Br}\right] \tag{2.29}$$

[2] For example, 310 K air at $u_b = 15$ m/s with $H = 1$ mm and $\Delta T = 10$ K would have: $\mathrm{Re}_{D_h} = 1797$, $\mathrm{Br} = 0.0158$, and $a\mathrm{Br}\Delta T = -0.122$ K.

For small Brinkman number, the third term (from dissipation) will be less than 5% of the second term (from the wall temperature change) until a sufficiently large x^+ is reached:

$$0.4G_0 \exp\left(-32\lambda_0^2 x^+/3\right) \geq 12|\mathrm{Br}| \tag{2.30}$$

Further, a calculation shows that the one-term approximation to the Graetz heat flux is within 5% of the full Graetz series for $x^+ \geq x_{qe}^+ = 0.008117$. Combining the two requirements,

$$x_{qe}^+ = 0.008117 \leq x^+ \leq -\frac{3}{32\lambda_0^2} \ln\left(\frac{30|\mathrm{Br}|}{G_0}\right) \equiv L_{\max}^+ \tag{2.31}$$

Simplifying to numerical values, the criterion is

$$0.008117 \leq x^+ < -0.1178 - 0.03315 \ln|\mathrm{Br}| \tag{2.32}$$

When the upper limit of Eq. (2.32) is satisfied, viscous dissipation accounts for less than 5% of the total heat flux. When the lower limit is also satisfied, the one-term (thermally developed) approximation to the flux is within 5% of the full Graetz series, Eq. (2.23).

So, what does Eq. (2.32) mean in a representative situation? Laminar flow is required, so we are limited to lowish speeds and smallish channels. Consider water (mean temp. 310 K, Pr = 4.63) with a 1 mm plate spacing at a relatively high velocity of 0.7 m/s. The Reynolds number is $\mathrm{Re}_{D_h} = 2005$, and if the temperature difference is +10 K, $\mathrm{Br} = 5.43 \times 10^{-5}$. Equation (2.32) gives $L_{\max}/D_h = 1929$. At that point, $\theta_b = 1.77 \times 10^{-3}$, i.e., 99.8% of the sensible heat associated with the wall temperature change has been removed before dissipation contributes even 5% to the heat flux. The bulk temperature rise from dissipation is $T_{d,b} = a\,\mathrm{Br}\Delta T = 0.37$ mK.[3]

It seems clear that dissipation can be ignored in many common scenarios.

Let us also consider a liquid that is quite viscous: glycerol at 313 K. For 5 mm spacing, $u_b = 0.5$ m/s, and a +10 K temperature change, we have: $\mathrm{Re}_{D_h} = 22$, Pr = 2451, and Br = 0.0249. Here, $L_{\max}^+ = 0.00464$. The high Prandtl number has a strong impact, so that $L_{\max}/D_h = 250$, or $L_{\max} = 2.5$ m, at which point $\theta_b = 0.809$. Dissipation becomes important in the region where the bulk temperature is still adjusting to the wall temperature change. The pressure drop over $L_{\max}$ is large, 1.7 atm, but the added temperature rise from dissipation is only $T_{d,b} = 0.171$ K.

In some situations, a higher pressure drop may be acceptable. If the glycerol in the previous example had $u_b = 5$ m/s, then $\mathrm{Re}_D = 220$, Br = 2.49, and $T_d = 17.1$ K, with a pressure drop of 17 atm. Dissipation is certainly not negligible in cases like this one.

The impact of nonzero Brinkman number on the local heat flux is interesting. The locations at which the bulk temperature of the Graetz solution is within 5%, 2%, and 1% of the wall temperature are indicated, along with the locations x_{qe}^+ and $L_{\max}^+$ at which the flux exceeds the thermally developed Graetz value by 5% as a result of entry length effects or dissipation, respectively. Between these locations, the thermally developed Graetz solution provides the heat flux to better than 5% accuracy. Since Br is often very small, we see that for many practical purposes, the Graetz solution may be used over the entire range of the response to a change in wall temperature.

[3] Perhaps there's an application in which a temperature difference below the level of measurement uncertainty remains important, but I can't immediately think of one.

Table 2.6: Brinkman number at which liquid dissipation makes less than a 5% or 1% contribution to θ_b or q_w from the Graetz solution at the points where θ_b equals 0.05, 0.02, or 0.01.

			|Br|			
θ_b	x^+	$q_w D_h/k\Delta T$	5% of θ_b	1% of θ_b	5% of q_w	1% of q_w
0.05	0.09621	0.3770	3.646×10^{-3}	7.292×10^{-4}	1.571×10^{-3}	3.142×10^{-4}
0.02	0.1266	0.1508	1.458×10^{-3}	2.917×10^{-4}	6.284×10^{-4}	1.257×10^{-4}
0.01	0.1496	0.07541	7.292×10^{-4}	1.458×10^{-4}	3.142×10^{-4}	6.284×10^{-5}

Table 2.6 shows the Brinkman number at which liquid dissipation makes less than a 5% or 1% contribution to θ_b or q_w as calculated from the Graetz solution at the location where θ_b equals 0.05, 0.02, or 0.01. Although the relative contribution of dissipation will be larger at values of x^+ greater than shown in the table, the amount of heat transferred upstream of the location shown accounts for the almost the entire heat load associated with the wall temperature change. Consequently, for practical purposes, dissipation may be ignored when $\mathrm{Br} \lesssim 10^{-3}$.

2.9 A problematic definition of the Nusselt number

References defined the heat transfer coefficient to include dissipation and flow work, using Eqs. (2.26) and (2.28) to evaluate $h = q_w/(T_b - T_w)$. The resulting Nusselt number is

$$\mathrm{Nu}_{D_h} = \frac{q_w D_h}{k(T_0 - T_w)\theta_b} \tag{2.33a}$$

$$= \frac{8\sum_{n=0}^{\infty} G_n \exp\left[-(32/3)\lambda_n^2 x^+\right] + 12\mathrm{Br}}{3\sum_{n=0}^{\infty}(G_n/\lambda_n^2)\exp\left[-(32/3)\lambda_n^2 x^+\right] + a\mathrm{Br}} \tag{2.33b}$$

where the second term in the numerator appears only for liquids.

Equation (2.33b) can be plotted for liquids with $\mathrm{Br} > 0$. In the region between the thermal entry length, x_{te}, and the location $L^{+,\mathrm{Nu}}_{\mathrm{max,Br}}$, this Nusselt number is within 5% of the thermally-developed Graetz value, 7.541. For reference, at $x^+ = 0.05$, $\theta_b = 0.202$ so that 80% of the heat transfer caused by wall temperature change has already occurred. For $x^+ = 0.1$, $\theta_b = 0.0446$, and 95.5% of that heat transfer has already occurred.

For $x^+ > L^{+,\mathrm{Nu}}_{\mathrm{max,Br}}$, this Nusselt number rises; and as $x^+ \to \infty$, $\mathrm{Nu}_{D_h} \to 35/2$ for liquids . In this final asymptotic region, the primary heat transfer process caused by the wall temperature change has entirely ended. Only heat flow from dissipation remains. As seen in Section 2.8, dissipation often causes only negligible heat flux and temperature difference. Equation (2.33b) goes to zero for gases as $x^+ \to \infty$.

Equation (2.33a) is problematic. For negative Brinkman number (the liquid heating case, the numerator changes sign at one point; and, at another point, the denominator passes through zero, causing a singularity! The reason for this bizarre behavior is that the baseline

of viscous heating has not been separated from the heat transfer induced by the wall temperature change when defining the convective heat transfer coefficient.

2.10 Superposition provides a better definition of h

In high-speed external boundary layers, viscous dissipation can be strong, and an adiabatic wall can be much hotter than the free stream flow. Whether heat flows into or out of an isothermal wall depends on the difference between the actual wall temperature and the adiabatic temperature, so that $q_w = h(T_{aw} - T_w)$. Superposition is used to separate the viscous heating of an adiabatic wall from the remaining convective heat flow, and analysis shows h to be the same as for flow without dissipation.

For an internal flow with an isothermal wall, on the other hand, the adiabatic wall temperature concept is not directly applicable, since the bulk temperature associated with dissipation would rise continuously along the length of an adiabatic channel. However, Eq. (2.28) has already shown us how to separate the dissipative heat transfer in liquids from the ordinary convective transport. We may write that equation in terms of the heat transfer coefficient from the Graetz solution

$$q_w(x^+) = q_{w,pp}(x^+) + \frac{6k(T_0 - T_w)}{H}\,\mathrm{Br} \tag{2.34a}$$

$$= h_{pp}(T_{pp,b} - T_w) + \frac{6k(T_0 - T_w)}{H}\,\mathrm{Br} \tag{2.34b}$$

$$= h_{pp}(T_{pp,b} - T_w) + \frac{6\mu u_b^2}{H} \tag{2.34c}$$

where the last right-hand side term is present only for liquids. Here, the heat transfer coefficient, $h_{pp} = q_{w,pp}/(T_{pp,b} - T_w)$, can be found from the Nusselt number of the Graetz solution in the usual way:

$$\mathrm{Nu}_{D_h} = \frac{h_{pp}D_h}{k} = \frac{8\sum_{n=0}^{\infty} G_n \exp\left[-(32/3)\lambda_n^2 x^+\right]}{3\sum_{n=0}^{\infty}(G_n/\lambda_n^2)\exp\left[-(32/3)\lambda_n^2 x^+\right]} \tag{2.35}$$

Flow work and/or dissipation move the bulk temperature above or below $T_{pp,b}$ by an increment $T_{d,b} = (T_0 - T_w)a\mathrm{Br} = a\mu u_b^2/k$

$$T_b = T_{pp,b} + a\mu u_b^2/k \tag{2.36}$$

where a was defined in Eq. (2.18a). As shown in Section 2.8, this increment is typically quite small and is often negligible.

Example. For the glycerol flow described in Section 2.8, let $u_b = 0.5$ m/s, $T_0 = 318$ K, and $T_w = 308$ K. At the location mentioned, 2.5 m after the wall temperature change, the flow is thermally developed with a Nusselt number of 7.541, giving $h_{pp} = 214.9$ W/m^{2}K. Here, $x^+ = 0.00464$, so that $\theta_b = 0.7915$ from Eq. (2.22), and $T_{pp,b} = 315.92$ K. The

Graetz heat flux is $q_{pp,w} = 214.9(315.92 - 308) = 1701$ W/m^{2}K. Finally, we may add the dissipative increments, $T_{d,b} = 0.171$ K and $q_d = 85.06$ W/m^{2}K, to get $T_b = 316.09$ K and $q_w = 1786$ W/m^{2}K.

Although the dissipative increase in q_w is +5% in this example, in many case it will be negligible. Note that the same numerical results are obtained using Eqs. (2.27) and (2.33b).

2.11 The Seban-Shimazaki criterion is misleading when dissipation is included

Seban and Shimazaki suggested that thermally developed internal flow must have a temperature profile shape that is independent of x, a criterion they expressed as

$$\frac{\partial}{\partial x}\left(\frac{T_w - T}{T_w - T_b}\right) = 0 \tag{2.37}$$

This equation is applied as a short-cut for calculating the heat transfer coefficient without evaluating heat transfer in the entry length . It ensures that analytical solutions for thermally developed flow will reproduce the empirically-observed constant heat transfer coefficient. The criterion does not appear to have any other basis in theory.

On the other hand, the profile shape of the Graetz solution (without dissipation) is inherently independent of x^+ in the thermally developed region where the one-term approximation applies:

$$\left(\frac{T_w - T_{pp}}{T_w - T_b}\right) = \left[\frac{B_0 \exp\left(-32\lambda_0^2 x^+/3\right) Y_0(\bar{y}) + \sum_{n=1}^{\infty} B_n \exp\left(-32\lambda_0^2 x^+/3\right) Y_n(\bar{y})}{3(G_0/\lambda_0^2)\exp\left(-32\lambda_0^2 x^+/3\right) + 3\sum_{n=1}^{\infty}(G_n/\lambda_n^2)\exp\left(-32\lambda_0^2 x^+/3\right)}\right] \tag{2.38}$$

$$\simeq \left(\frac{\lambda_0^2 B_0 Y_0(\bar{y})}{3G_0}\right) \text{ for } x^+ > x_{te}^+ \tag{2.39}$$

In that light, the Seban-Shimazaki criterion is simply a characteristic of the entry length solution in the one-term limit. It is not a constraint to be applied in finding the solution.[4]

Seban and Shimazaki studied fully-developed turbulent flow. The classical analysis of the turbulent thermal entry length also results in a Fourier series with a profile shape independent of x^+ in the one-term limit . Again, the criterion just describes a feature of the one-term limit.

When flow work and/or dissipation are present, the profile shape behaves differently in the one-term limit. With Eqs. (2.20), (2.26), and (2.27b), the one-term limit of the shape is

$$\left(\frac{T_w - T}{T_w - T_b}\right) \simeq \left(\frac{B_0 \exp\left(-\frac{32}{3}\lambda_n^2 x^+\right) Y_0(\bar{y}) + \mathrm{Br}\, f(\bar{y})}{3(G_0/\lambda_0^2)\exp\left(-\frac{32}{3}\lambda_n^2 x^+\right) + a\mathrm{Br}}\right) \tag{2.40}$$

[4]To view it otherwise is to adopt the fallacy of logic: *post hoc, ergo propter hoc* ("After this, therefore because of this").

Equation (2.40) is approximately independent of x^+ in the range $x_{te}^+ \leq x^+ \leq L_{\text{max, Br}}^{+,\text{Nu}}$, but only fully independent when $x^+ \to \infty$. That asymptotic condition is a steady-state of viscous dissipation, which appears only after the heat transfer caused by the wall temperature change has ended. The same condition is present upstream of the wall temperature change. This state provides no information about the heat transfer caused by a wall temperature change.

The Seban-Shimazaki criterion is therefore misleading when dissipation is included if we aim to compute the response to a wall temperature change—as we do in heat exchangers and many other practical scenarios.

Paolucci used Eq. (2.37) to define thermally developed flow and chose h as in Section 2.9, thus setting his thermally developed solutions to the asymptotic limit. Ou and Cheng did not use Eq. (2.37), and they provided graphical results over the entire range of x^+. However, they took $x \to \infty$ as the final state and concluded that 7.541 is an incorrect value for thermally developed flow, writing "dissipation cannot be neglected under any circumstances in the case of uniform wall temperature." As shown herein, such "neglect" can be quite appropriate in practical scenarios.

The essential feature of analytical solutions for thermally developed flow is that a one-term approximation of the Fourier series is sufficient for large values of x^+. If dissipation is negligible, these one-term solutions will meet the Seban-Shimazaki criterion. A similar behavior occurs in the one-term approximations commonly used for transient heat conduction at large Fourier number .

2.12 Summary

The classical Graetz solution for isothermal plates, reported in most textbooks, is perfectly acceptable because flow work and/or dissipation effects will be negligible in most situations of engineering relevance. In particular, for ideal gases, an accurate calculation of the heat flux can be made while entirely ignoring these effects. In a liquid, one may wish to test the neglect of dissipation, in which case Eq. (2.32) provides an estimate of the range within which the thermally developed Graetz heat flux will be accurate. The thermally developed Nusselt number for laminar flow between isothermal parallel plates is accurately represented as 7.541.

Reports that flow work and/or dissipation radically change the thermally developed Nusselt number stem from rigid adherence to definitions without adequate consideration of the purpose and meaning of the definitions. A more useful definition of the heat transfer coefficient (or Nusselt number) should remove the dissipative wall heat flux and the dissipative contribution to bulk temperature. When that is done, the result is the well-known Nusselt number of the Graetz solution. Dissipative effects, if non-negligible, are not a function of x^+ and may be added to the solution separately. This approach avoids creating a Nusselt number that depends sensitively on the Brinkman number and which can have a singularity.

The Seban-Shimazaki criterion for thermally developed flow is a convenient shortcut to the thermally developed result: a heat transfer coefficient that becomes constant downstream. The criterion is misleading when dissipation is present. A more practical

criterion in this case defines thermally developed flow as the one-term approximation to the series solution. If dissipation is neglected, the Seban-Shimazaki criterion merely summarizes a property of the one-term approximation.

I have not discussed channel cross-sections other than parallel plates, and my analysis of operating conditions is limited. Further work could more comprehensively map out the regimes in which dissipation is negligible. However, a tiny Brinkman number, say $\lesssim 10^{-3}$, is itself a strong indicator that dissipative effects are not significant.

Nomenclature for Chapter 2

Roman letters

a	constant defined in Eq. (2.18a)
B_n	coefficient in series, Eq. (2.20)
Br	Brinkman number, $\mu u_b^2/k(T_0 - T_w)$
c_p	isobaric specific heat capacity [J kg^{-1} K^{-1}]
D_h	hydraulic diameter, $2H$ [m]
d_{ij}	deviatoric stress tensor [Pa]
$f(\bar{y})$	see Eq. (2.17b)
G_n	coefficient in series, Eq. (2.21), Table 1.1
H	plate separation [m]
k	thermal conductivity [W m^{-1} K^{-1}]
L	length of channel [m]
L^+_{max}	$L_{\text{max}}/(\text{Re}_{D_h}\,\text{Pr}\,D_h)$
Nu_{D_h}	local Nusselt number, Eq. (2.35)
$u(y)$	fully-developed velocity profile [m s^{-1}]
u_b	bulk velocity [m s^{-1}]
$\bar{u}$	dimensionless velocity, $u(y)/u_b$
p	pressure [Pa]
Pr	Prandtl number, $\mu c_p/k$
q_w	heat flux into wall [W m^{-2}]
Re_{D_h}	Reynolds number, $\rho u_b D_h/\mu$
T	temperature [K]
T_0	wall temperature for $x < 0$ [K]
T_{aw}	adiabatic wall temperature [K]
T_w	wall temperature for $x > 0$ [K]
x	streamwise coordinate [m]
x^+	$x/(\text{Re}_{D_h}\,\text{Pr}\,D_h)$
x^+_{qe}	position for q_w within 5% of thermally-developed value
x^+_{te}	thermal entry length of Graetz solution
$Y_n(\bar{y})$	eigenfunction, Eq. (2.20)
y	cross-stream coordinate [m]
$\bar{y}$	dimensionless coordinate, y/H

Greek letters

β	thermal expansion coefficient [K^{-1}]
ΔT	$T_0 - T_w$ [K]
θ	nondimensional temperature
λ_n	eigenvalue, Eq. (2.20), Table 1.1
μ	dynamic viscosity [kg/m-s]
ρ	density [kg m^{-3}]

Superscripts and subscripts

b	bulk value
d	case with flow work and/or dissipation only
nd	case without flow work and/or dissipation
max	maximum length to be < 5% above thermally developed value
Nu	5% limit for the Nusselt number
pp	Graetz solution for parallel plates

Appendix A

Multipage and sideways tables

A.1 A multipage table of numbers

This example uses the `longtable` package. The values in the table are the one-term coefficients for one-dimensional, transient heat conduction, $\theta = A_1 f_1 \exp(-\lambda_1^2 \mathrm{Fo})$, $\overline{\theta} = D_1 \exp(-\lambda_1^2 \mathrm{Fo})$.

Table A.1: One-term coefficients for one-dimensional heat conduction with a convective boundary condition. Data follow H. D. Baehr and K. Stephan [21].

Bi	*Plate*			*Cylinder*			*Sphere*		
	λ_1	A_1	D_1	λ_1	A_1	D_1	λ_1	A_1	D_1
0.01	0.09983	1.0017	1.0000	0.14124	1.0025	1.0000	0.17303	1.0030	1.0000
0.02	0.14095	1.0033	1.0000	0.19950	1.0050	1.0000	0.24446	1.0060	1.0000
0.03	0.17234	1.0049	1.0000	0.24403	1.0075	1.0000	0.29910	1.0090	1.0000
0.04	0.19868	1.0066	1.0000	0.28143	1.0099	1.0000	0.34503	1.0120	1.0000
0.05	0.22176	1.0082	0.9999	0.31426	1.0124	0.9999	0.38537	1.0150	1.0000
0.06	0.24253	1.0098	0.9999	0.34383	1.0148	0.9999	0.42173	1.0179	0.9999
0.07	0.26153	1.0114	0.9999	0.37092	1.0173	0.9999	0.45506	1.0209	0.9999
0.08	0.27913	1.0130	0.9999	0.39603	1.0197	0.9999	0.48600	1.0239	0.9999
0.09	0.29557	1.0145	0.9998	0.41954	1.0222	0.9998	0.51497	1.0268	0.9999
0.10	0.31105	1.0161	0.9998	0.44168	1.0246	0.9998	0.54228	1.0298	0.9998
0.15	0.37788	1.0237	0.9995	0.53761	1.0365	0.9995	0.66086	1.0445	0.9996
0.20	0.43284	1.0311	0.9992	0.61697	1.0483	0.9992	0.75931	1.0592	0.9993
0.25	0.48009	1.0382	0.9988	0.68559	1.0598	0.9988	0.84473	1.0737	0.9990
0.30	0.52179	1.0450	0.9983	0.74646	1.0712	0.9983	0.92079	1.0880	0.9985
0.40	0.59324	1.0580	0.9971	0.85158	1.0931	0.9970	1.05279	1.1164	0.9974
0.50	0.65327	1.0701	0.9956	0.94077	1.1143	0.9954	1.16556	1.1441	0.9960
0.60	0.70507	1.0814	0.9940	1.01844	1.1345	0.9936	1.26440	1.1713	0.9944
0.70	0.75056	1.0918	0.9922	1.08725	1.1539	0.9916	1.35252	1.1978	0.9925
0.80	0.79103	1.1016	0.9903	1.14897	1.1724	0.9893	1.43203	1.2236	0.9904
0.90	0.82740	1.1107	0.9882	1.20484	1.1902	0.9869	1.50442	1.2488	0.9880
1.00	0.86033	1.1191	0.9861	1.25578	1.2071	0.9843	1.57080	1.2732	0.9855
1.10	0.89035	1.1270	0.9839	1.30251	1.2232	0.9815	1.63199	1.2970	0.9828
1.20	0.91785	1.1344	0.9817	1.34558	1.2387	0.9787	1.68868	1.3201	0.9800
1.30	0.94316	1.1412	0.9794	1.38543	1.2533	0.9757	1.74140	1.3424	0.9770

Table A.1: (continued)

Bi	*Plate*			*Cylinder*			*Sphere*		
	λ_1	A_1	D_1	λ_1	A_1	D_1	λ_1	A_1	D_1
1.40	0.96655	1.1477	0.9771	1.42246	1.2673	0.9727	1.79058	1.3640	0.9739
1.50	0.98824	1.1537	0.9748	1.45695	1.2807	0.9696	1.83660	1.3850	0.9707
1.60	1.00842	1.1593	0.9726	1.48917	1.2934	0.9665	1.87976	1.4052	0.9674
1.70	1.02725	1.1645	0.9703	1.51936	1.3055	0.9633	1.92035	1.4247	0.9640
1.80	1.04486	1.1695	0.9680	1.54769	1.3170	0.9601	1.95857	1.4436	0.9605
1.90	1.06136	1.1741	0.9658	1.57434	1.3279	0.9569	1.99465	1.4618	0.9570
2.00	1.07687	1.1785	0.9635	1.59945	1.3384	0.9537	2.02876	1.4793	0.9534
2.20	1.10524	1.1864	0.9592	1.64557	1.3578	0.9472	2.09166	1.5125	0.9462
2.40	1.13056	1.1934	0.9549	1.68691	1.3754	0.9408	2.14834	1.5433	0.9389
2.60	1.15330	1.1997	0.9509	1.72418	1.3914	0.9345	2.19967	1.5718	0.9316
2.80	1.17383	1.2052	0.9469	1.75794	1.4059	0.9284	2.24633	1.5982	0.9243
3.00	1.19246	1.2102	0.9431	1.78866	1.4191	0.9224	2.28893	1.6227	0.9171
3.50	1.23227	1.2206	0.9343	1.85449	1.4473	0.9081	2.38064	1.6761	0.8995
4.00	1.26459	1.2287	0.9264	1.90808	1.4698	0.8950	2.45564	1.7202	0.8830
4.50	1.29134	1.2351	0.9193	1.95248	1.4880	0.8830	2.51795	1.7567	0.8675
5.00	1.31384	1.2402	0.9130	1.98981	1.5029	0.8721	2.57043	1.7870	0.8533
6.00	1.34955	1.2479	0.9021	2.04901	1.5253	0.8532	2.65366	1.8338	0.8281
7.00	1.37662	1.2532	0.8932	2.09373	1.5411	0.8375	2.71646	1.8673	0.8069
8.00	1.39782	1.2570	0.8858	2.12864	1.5526	0.8244	2.76536	1.8920	0.7889
9.00	1.41487	1.2598	0.8796	2.15661	1.5611	0.8133	2.80443	1.9106	0.7737
10.00	1.42887	1.2620	0.8743	2.17950	1.5677	0.8039	2.83630	1.9249	0.7607
12.00	1.45050	1.2650	0.8658	2.21468	1.5769	0.7887	2.88509	1.9450	0.7397
14.00	1.46643	1.2669	0.8592	2.24044	1.5828	0.7770	2.92060	1.9581	0.7236
16.00	1.47864	1.2683	0.8541	2.26008	1.5869	0.7678	2.94756	1.9670	0.7109
18.00	1.48830	1.2692	0.8499	2.27556	1.5898	0.7603	2.96871	1.9734	0.7007
20.00	1.49613	1.2699	0.8464	2.28805	1.5919	0.7542	2.98572	1.9781	0.6922
25.00	1.51045	1.2710	0.8400	2.31080	1.5954	0.7427	3.01656	1.9856	0.6766
30.00	1.52017	1.2717	0.8355	2.32614	1.5973	0.7348	3.03724	1.9898	0.6658
35.00	1.52719	1.2721	0.8322	2.33719	1.5985	0.7290	3.05207	1.9924	0.6579
40.00	1.53250	1.2723	0.8296	2.34552	1.5993	0.7246	3.06321	1.9942	0.6519
50.00	1.54001	1.2727	0.8260	2.35724	1.6002	0.7183	3.07884	1.9962	0.6434
60.00	1.54505	1.2728	0.8235	2.36510	1.6007	0.7140	3.08928	1.9974	0.6376
80.00	1.55141	1.2730	0.8204	2.37496	1.6013	0.7085	3.10234	1.9985	0.6303
100.00	1.55525	1.2731	0.8185	2.38090	1.6015	0.7052	3.11019	1.9990	0.6259
200.00	1.56298	1.2732	0.8146	2.39283	1.6019	0.6985	3.12589	1.9998	0.6170
∞	1.57080	1.2732	0.8106	2.40483	1.6020	0.6917	3.14159	2.0000	0.6079

A.2 A sideways table

Table A.2 uses the `sidewaystable*` environment from the `rotating` package.

Table A.2: Data sets for local heat transfer coefficients in liquids. Boundary condition (B.C.) and r.m.s. freestream turbulence (u_r'/u_∞) are as indicated.

Authors	B.C.	Range of Re_x	u_r'/u_∞	Comments
Data for Water				
Hollingsworth, 1989	UWT	$\approx 5\times10^5$	0.5 – 1.5%	Data for $\mathrm{Pr} = 5.93 \pm 0.13$ with 4 K b.l. temperature difference. Boundary layer tripped ahead of plate, with a substantial unheated section; two locations measured in uncurved channel. Plate temperature controlled by rows of electrically heated strips. Data analysis focuses on enthalpy thickness Reynolds number. Data were used to calibrate a numerical model, leading to a correlation: $\mathrm{St} = 0.02426\ \mathrm{Re}_x^{-n}\mathrm{Pr}^{-0.895}$ with $n = 0.1879\ \mathrm{Pr}^{-0.18}$ for $0.7 \leqslant \mathrm{Pr} \leqslant 8$. Measurement uncertainties (2σ): St ± 3.3%; Re_x ± 1.6%.
Žukauskas & Šlančiauskas, 1987 summary	UHF	4.0×10^3 – 4.4×10^6	~ 1.2%	Data for Prandtl numbers of 2.95, 5.4, and 6.6. Stated uncertainty in heat transfer is up to ±10%, being largest for higher fluxes. Background turbulence measured at inlet to test section. For Pr = 5.4: Run 17, ΔT is 7–8 K for the turbulent and transitional data, with the laminar points rising to 17 K; Run 18, $6.1 \leqslant \Delta T \leqslant 7.5$ K; Table 25, ΔT is 7–8 K for the turbulent and transitional data, again rising to 15–17 K in the laminar region. For Pr = 6.6: Run 7, $1.3 \leqslant \Delta T \leqslant 4.2$ K; Run 8, $6.5 \leqslant \Delta T \leqslant 12.5$ K. For Pr = 2.95: $5.4 \leqslant \Delta T \leqslant 8.6$ K.
Data for higher Pr				
Žukauskas & Šlančiauskas, 1987 summary	UHF	1.2×10^4 – 7.7×10^5	~ 0.7%	Data for "transformer oil" at several Prandtl numbers (55–257). Uncertainty in heat transfer stated as ±5%. Background turbulence measured at inlet to test section. Many of these data embed significant property variation across the boundary layer. (An additional data set is given for cooling hot glycerin at higher Pr, but with even stronger property variations.)

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